# The Tennis Ball Paradox

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On several occasions I have been told the paradox about a room and some tennis balls. The story goes that in the first half minute two tennis balls, #1 and #2, are tossed into the room and #1 is thrown out. In the next quarter minute balls #3 and #4 are tossed in and #2 is tossed out. In the next $1/8$ minute #5 and #6 are tossed in and #3 is tossed out. Etc.

1. I was told that in the limit there are no tennis balls in the room because "if you think that there are then when you name the ball I can tell you when it was thrown out."

2. However, I said to myself that if I look at the number of balls in the room, then at stage $N$ there are $N$ balls left and I am being asked to believe that the limit of $N$ as $N$ goes to infinity is 0.

To get a different view on what is essentially the same situation we examine a well-known mathematical result. The constant $\gamma$ is defined by

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{n}-\log N\right\}=\gamma=0.5772\ 15664\ldots.$$

Define

$$\gamma_N=\sum_{n=1}^{N}\frac{1}{n}-\log N \qquad (\gamma_N\to\gamma)$$

and

$$\begin{aligned} S_N=\gamma_{2N}-\gamma_N &= \sum_{n=N+1}^{2N}\frac{1}{n}-\log\frac{2N}{N} \\ &= \left[\frac{1}{N+1}+\frac{1}{N+2}+\cdots+\frac{1}{2N}\right]-\log 2. \end{aligned}$$

Since $\gamma_{2n}$ and $\gamma_n$ both approach $\gamma$,

$$\lim_{N\to\infty}\left[\frac{1}{N+1}+\frac{1}{N+2}+\cdots+\frac{1}{2N}\right]=\log 2.$$

We see that, as in the tennis ball paradox, at each stage (as $N$ goes to $N+1$) two terms of the series are added and one term is removed. The first argument would assert that in the limit there are no terms in the sum since if you name the particular term, then I can tell you when it was removed; hence the sum, so the argument goes, has no terms. Yet the sum is apparently $\log 2$. The difference between the two examples is that in the series we have the terms $1/n$, and in the tennis balls we count the number by integers and each corresponding term is 1. Also, the limit of the tennis balls is infinite while for the second illustration the limit is finite.

The trouble arises from the introduction of dynamics into mathematics—which is common in thinking about limits! If we use the terminology of the standard textbook for handling limits then we have the two situations:

1. For any given term $N$ there exists an $n_0$ such that for all $n \geqslant n_0$ the ball #$N$ is not there.

2. For any given $N$ there exists an $n_0$ such that for all $n \geqslant n_0$ the number of balls in the room is greater than $N$.

It is the translation into colorful words that produces the paradox; one doubts the wisdom of the first translation to the assertion that in the limit there are no balls in the room. The careful textbook methods were developed for clarity in thinking, not just to befuddle the student.

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# Unexpected Occurrences of the Number $e$

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Martin Gardner once observed [2, p. 34] that of the three most famous irrational numbers, $\pi$, the golden ratio, and $e$, the third is least familiar to students early in their study of mathematics. The number $e$, named by Leonhard Euler (1707–1783), is usually encountered for the first time during the second course in calculus, either through the equation

$$\int_1^e \frac{dt}{t} = 1,$$

introduced in connection with natural logarithms, or else through the equation

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

However, neither of these definitions of $e$ provides immediate insight into this important number. As a result, students rarely come away with a solid grasp of this number, beyond, perhaps, rote memory of the phrase, "It is the base of the natural logarithms." Given the importance of $e$, students' first meeting with this number should be a more memorable experience than the standard introductions provide.

In this note we present an expository catalog of occurrences of $e$ in probability which might be used to increase students' appreciation of this number. Most likely, some, but not all, of these examples will be familiar to teachers of mathematics.

**Example 1.** Each of two people is given a shuffled deck of playing cards. Simultaneously they expose their first cards. If these cards do not match (for example, two "four of clubs" would be considered a "match"), they proceed to expose their second cards and so forth through the decks. What is the probability of getting through the decks without a single match? In [4, p. 281] it is shown that the answer is given by the sum,

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{52!},$$

which is the initial portion of a series for $1/e$ (based on the Maclaurin series