Transits of Venus and the Astronomical Unit

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Edmond Halley. Leonhard Euler. Mason and Dixon (of "line" fame). Captain James Cook. James Short.

With the probable exception of the last man listed, these names are all very recognizable, yet it seems most unusual for them to appear together. What could they possibly have in common?

The answer, of course, is given away in the title of this article. All were players in the extraordinary story surrounding observations of the transits of Venus—that is, the passages of Venus across the disk of the Sun, as viewed from Earth—that took place in the eighteenth century. Only five transits of Venus are known to have been observed in the history of mankind, in 1639, 1761, 1769, 1874, and 1882. Thus no one alive today has seen one. But this will soon change, for the next transit will take place June 8, 2004, and another will follow on June 6, 2012.

Though transits of Venus are rare and beautiful astronomical events, they could not have earned a significant place in the history of science for aesthetic reasons alone. *The extraordinary attention devoted to these transits, especially in 1761 and 1769, was due to their usefulness in determining the length of the astronomical unit, that is, the mean distance from Earth to the Sun, in terms of terrestrial distance units such as miles.* Indeed, one estimate of the astronomical unit, computed from observations of the 1769 transit and published in 1771, differs from modern radar-based values by a mere eight-tenths of a percent [**5, 9**].

The first purpose of this article is to offer a glimpse into the rich history surrounding observations of the transits of Venus, especially the transit of 1761. But a second and more important purpose is to give a mathematical description of the methods used by Mr. James Short following the 1761 transit to deduce the length of the astronomical unit. As June 8, 2004 draws near, one is sure to read of the upcoming transit in the popular press. This article is intended to augment the popular accounts by providing mathematical insight into the event for those who are able to appreciate it.

Kepler's prediction and the first observed transits

Our story starts with the German astronomer Johannes Kepler in the early part of the seventeenth century. Though Kepler never witnessed a transit himself, his significance in the story is enormous for two reasons.

First, according to Kepler's Third Law, as it is now known, the ratio of the square of a planet's orbital period to the cube of its mean distance from the Sun is the same for all planets. From this law, the *relative* scale of the solar system can be determined simply by observing the orbital periods of the planets. In fact, Kepler's own estimates of the relative distances of the known planets from the Sun do not differ significantly from modern values. But Kepler was unable to translate his discovery of the relative scale of the solar system into absolute terms, for he badly underestimated the length of the astronomical unit. His estimate of 3469 Earth radii (actually the largest of several of his estimates) was roughly seven times too small, and so his understanding of absolute

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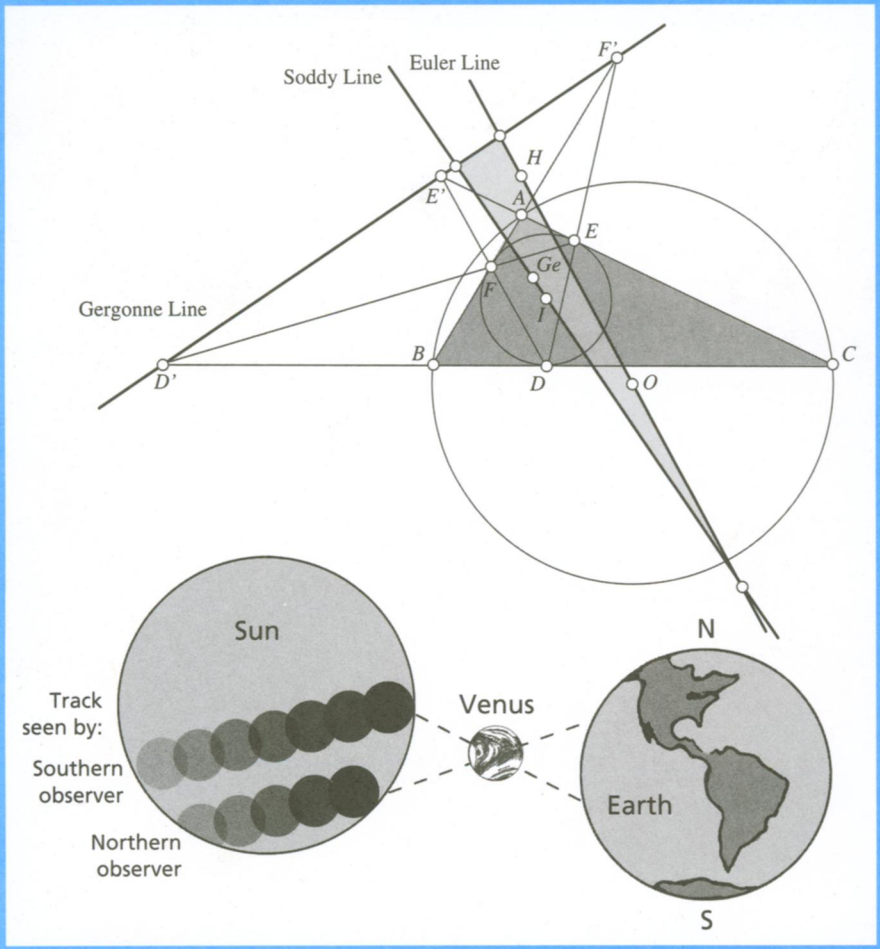

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distances within the solar system remained considerably flawed [**13**]. Despite Kepler's naive estimate of the solar distance, his third law remains one of the great achievements in the history of science, and is unquestionably fundamental to understanding the size of the solar system.

The second connection between Kepler and the transit problem is much more direct. It was his prediction in 1629 of the transits of Mercury, in November of 1631, and Venus, in December of 1631, that led to the first-ever observations of such events. Kepler predicted that the Venus transit would not be visible in Europe, nevertheless he asked astronomers to keep watch on the 6th and 7th of December in case his calculations were imperfect. He also "directed his request to observe this transit... to sailors who would be on the high seas, and learned men in America..." [**13**]. Unfortunately, there is no evidence that anyone successfully observed the 1631 Venus transit. On the other hand, at least three people saw the transit of Mercury in 1631 as a result of Kepler's prediction. Of these, Pierre Gassendi wrote a detailed account of the event. Though no attempt was made to use this transit to determine the length of the astronomical unit, Gassendi's observation was significant nonetheless, for it revealed that the apparent diameter of Mercury was far smaller than had been assumed by Kepler and his contemporaries [**7**]. Kepler, unfortunately, died on November 15, 1630, and thus did not live to see his brilliant prediction fulfilled.

Kepler's transit predictions were based on his *Rudolphine Tables* of 1627, which were produced as a result of his work with the great Danish astronomer Tycho Brahe. By the same method, Kepler also predicted the 1761 transit of Venus, but imperfections in the tables led him to believe that no transit would take place in 1639. Following Kepler's death, the Belgian astronomer Philip van Lansberg produced a set of tables, now known to be considerably inferior to Kepler's, but which did in fact predict a transit in 1639. It was in trying to reconcile differences between Lansberg's tables and the *Rudolphine Tables* that a brilliant young Englishman, Mr. Jeremiah Horrox, became convinced that a Venus transit would indeed occur in 1639 [**15**]. Regarding Lansberg's tables, Horrox wrote [**14**], "I pardon, in the meantime, the miserable arrogance of the Belgian astronomer, who has overloaded his useless tables with such unmerited praise... deeming it a sufficient reward that I was thereby led to consider and forsee the appearance of Venus in the Sun."

Horrox was richly rewarded for his labors in correcting the *Rudolphine Tables*, for on December 4, 1639, he became one of the first two people ever to observe a transit of Venus. The event was also observed at a nearby location by his friend William Crabtree, whom Horrox had alerted in the weeks preceding the event. No attempt was made by Horrox to use the Venus transit to determine the solar distance, but as with Gassendi's observations of Mercury in 1631, the event served to show that the angular size of Venus was far smaller than had been assumed [**13**].

Solar parallax

We now pause to present a few technical terms that are essential for the development of the story, using the terminology found in Taff [**12**]. In FIGURE 1, α is the angle between the line through the centers of the Earth and Sun and a line through the center

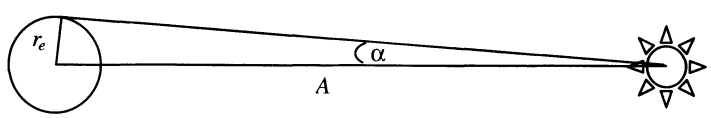


Figure 1 The equatorial horizontal solar parallax α

of the Sun and tangent to the Earth's surface. The angle α is known as the *horizontal solar parallax*. Denoting the Earth's equatorial radius by r_e and the Earth-Sun distance by A, it is clear from FIGURE 1 that $\alpha = \sin^{-1}(r_e/A)$. But of course A is not constant! If A is chosen to be one astronomical unit (au), the mean distance from Earth to the Sun, then α is known as the *mean equatorial horizontal solar parallax*. For brevity, the term *solar parallax* is commonly used in place of mean equatorial horizontal solar parallax, with further distinctions made only when the context requires it. With the preceding definitions and the understanding that r_e is known, it should be clear that *the problem of finding the length of the astronomical unit in terms of terrestrial units is equivalent to determining the solar parallax* α. Finally, to properly prepare the reader for the discussion that follows, we note that a modern value for the solar parallax is $8.794148''$ [**12**], where a minute ($'$) is the sixtieth part of a degree and a second ($''$) is the sixtieth part of a minute. We also point out that *smaller* estimates of the solar parallax correspond to *larger* estimates of the astronomical unit, as expressed in miles (or kilometers, or Earth radii, or...).

Kepler's Earth-Sun distance of 3469 Earth radii corresponds to a solar parallax of about one minute. Through the course of the seventeenth century, estimates of the solar parallax continued to diminish, due in large part to a vast increase in the quality and quantity of telescopic observations of the planets. By the end of the century, leading astronomers had all begun to believe that the solar parallax was considerably less than one minute, though there was little uniformity and often less than compelling reasoning behind the variety of values that continued to appear in scholarly works. The uncertainty that remained early in the eighteenth century is nicely illustrated in van Helden's *Measuring the Universe* [**13**], where we find that no less an authority than Newton was still undecided about the solar parallax: In the second edition of the *Principia* (1713), he used $10''$; in notes for the third edition he variously used $11''$, $12''$, and $13''$, and in the third edition itself one finds a solar parallax of $10\frac{1}{2}''$.

Edmond Halley's call for action: an international scientific effort

Though the idea of using a transit of Venus or Mercury to determine the solar parallax dates back at least to the Scottish mathematician James Gregory in 1663, it was Edmond Halley who became its greatest advocate. Halley observed a transit of Mercury from the southern hemisphere in 1677, and in his report on the observations, he discussed the possibility of using transits of Mercury or Venus to determine the solar parallax. Of the two, he believed that the geometry of Venus transits was far more likely to produce accurate results. Halley proposed the Venus transit idea in papers presented to the Royal Society in 1691, 1694, and most importantly, in 1716. Because Halley was one of the most influential astronomers of his time (he became the second Astronomer Royal in 1719), his paper of 1716 became "a clarion call for scientists everywhere to prepare for the rare opportunity presented by the forthcoming transits of 1761 and 1769." [**15**]

Halley's 1716 paper [**4**] begins by lamenting the wide variety of solar parallax values in use at the time, some as large as $15''$, and suggests $12\frac{1}{2}''$ as a plausible value. He goes on to describe roughly his method of determining the solar parallax from observations of the transit of Venus that would take place in 1761, even going so far as to describe the proper locations to send observers. "Therefore again and again," writes Halley, "I recommend it to the curious strenuously to apply themselves to this observation. By this means, the Sun's parallax may be discovered, to within its five hundredth part..." The essence of his method was to calculate, based on the $12\frac{1}{2}''$ hypothesis, the *expected* difference in the duration of the transit as observed at two widely differ-

ing locations. "And if this difference be found to be greater or less by observation, the Sun's parallax will be greater or less nearly in the same ratio." As we shall see, this was exactly the idea behind the methods employed by James Short when the transit actually took place. But despite the claim by Acker and Jaschek [**1**] that "this method was used by Halley in 1761 and 1769," Halley had no illusions that he would personally put his method into practice, for he died in 1742 at the age of 85.

Halley's paper called for observers to be stationed far and wide across the globe, a monumental task in 1761. Despite the obvious difficulties involved in sending observers to distant locations, not to mention the fact that Great Britain and France were in the midst of the Seven Years' War at the time, the response to his call was overwhelming. In all, when the transit took place, there were at least 122 observers at sixty-two separate stations, from Calcutta to the Siberian city of Tobolsk, from the Cape of Good Hope to St. John's in Newfoundland, and of course, at a large number of locations throughout Europe [**15**]. Many had traveled weeks or even months to reach their destinations. Unfortunately, it is impossible to describe in this short article all the adventures of those who set out to observe the 1761 transit: of Charles Mason and Jeremiah Dixon who set out for the East Indies, but hadn't so much as left the English channel when their ship was attacked by a French warship, leaving 11 dead and 37 wounded; of the Frenchman Chappe who traveled 1500 miles across Russia to Tobolsk by horse-drawn sleigh, once having to round up his deserting guides at gunpoint; of the Frenchman Le Gentil who was prevented by the war from reaching his destination in India, and so was forced to observe the transit from the rolling deck of a ship in the Indian Ocean. The interested reader will find excellent descriptions of these and other expeditions in Harry Woolf's book on the eighteenth-century transits of Venus [**15**]. All in all, the efforts to observe the 1761 transit of Venus surely amounted to the greatest international scientific collaboration in history up to that time.

James Short and his computation of the solar parallax

James Short (1710–1768) is not well known in modern mathematical circles for the simple reason that he was not primarily a mathematician. Though Short studied under Colin Maclaurin and displayed some talent in mathematics, he achieved fame and fortune as one of the most skilled telescope makers of the eighteenth century. In his lifetime, Short made some 1,370 telescopes, of which 110 still exist today [**3**]. A "Short biography" might also mention that he was a candidate for the post of Astronomer Royal, a frequent contributor to the *Philosophical Transactions of the Royal Society*, a friend of Benjamin Franklin, and a co-discoverer of a nonexistent moon of Venus [**3, 6**]. Short was a member of a special committee established by the Royal Society to plan the study of the 1769 transit of Venus, but died before the plan could be implemented.

Short observed the 1761 transit of Venus from London, in the company of the Duke of York and other honored guests. In the months following the transit, Short collected a good deal of data from the various observations that had taken place worldwide. These he published in the *Philosophical Transactions* in December 1761, in a paper entitled *The Observations of the internal Contact of Venus with the Sun's Limb, in the late Transit, made in different Places of Europe, compared with the Time of the Same Contact observed at the Cape of Good Hope, and the Parallax of the Sun from thence determined* [**10**]. A second article [**11**], virtually identical in nature but with a great deal more data, appeared a year later in an attempt to strengthen the case for his computed solar parallax value. We shall now examine the methods Short used, as described in the 1761 paper.

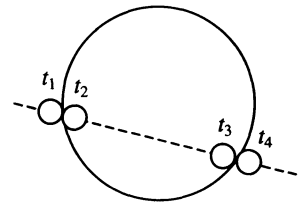


Figure 2 Contacts at ingress and egress

FIGURE 2 illustrates the positions of Venus on the disk of the Sun at four crucial times during the transit. Times t_1, t_2, t_3, and t_4 are the times of *external contact at ingress, internal contact at ingress, internal contact at egress*, and *external contact at egress*, respectively. Next, in FIGURE 3, one can see that the track of Venus across the Sun shifts *upward* as the observer moves further *south* on the surface of the Earth. This upward shift has two consequences that are crucial to Short's computational plans: first, the t_3 time is earlier for northern observers than for southern observers, and second, the total duration of the transit $t_3 - t_2$ is shorter for northern observers than for southern observers. Short's two methods simply amount to quantifying these two ideas.

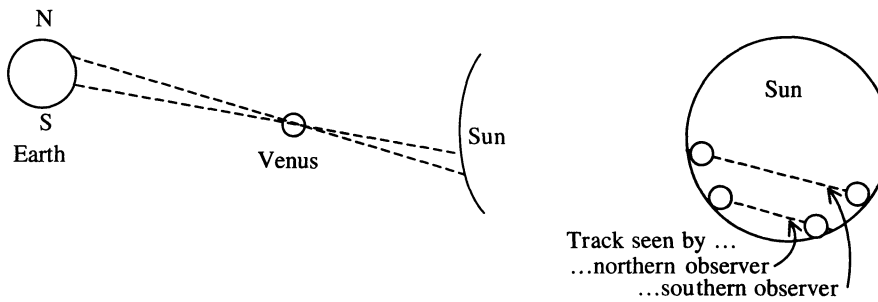


Figure 3 Effect of latitude on the apparent track of Venus

We can easily illustrate Short's first method just as he presents it in his first paper, that is, with virtually no computational details whatsoever! First, we note that the t_3 time observed in Greenwich was 8:19:00 AM local time, whereas the t_3 time as observed at the Cape of Good Hope was 9:39:50 local time. (In fact, it was Mason and Dixon who provided the valuable observations from the Cape, having been prevented from reaching the East Indies by their skirmish with the French warship.) The difference is $1^h\,20'\,50''$. Now most of this difference is due to the difference in local times, which Short determines to be $1^h\,13'\,35''$. Since one hour of local time difference corresponds to 15° of longitude, Short's figure is equivalent to saying that the Cape's longitude is 18°23′45″ east of Greenwich. But after the difference in local times is accounted for, a time difference of $7'\,15''$ remains, which must be the difference due to the effect of latitude illustrated in FIGURE 3.

Next, Short asserts a theoretical difference to compare with this observed difference of $7'\,15''$. *Assuming a solar parallax of* $8.5''$ *on the day of the transit, the* t_3 *time for an observer at the Cape should be* $6'\,8''$ *later than the* t_3 *time for a hypothetical observer at the center of the Earth, and the* t_3 *time for an observer at Greenwich should be* $1'\,11''$ *earlier*. Thus the $8.5''$ hypothesis leads to a difference of $7'\,19''$ between the t_3 times predicted for these two stations. "But the difference in absolute time," Short writes,

"as found by observation, as above, is only $= 7'\,15''$, therefore the Sun's parallax, by supposition, viz. $8.5''$, is to the parallax of the Sun found by observation, as $7'\,19''$ is to $7'\,15''$, which gives $8.42''$ for the Sun's parallax, on the day of the transit, by this observation..." In other words, after converting times to seconds, Short has solved the proportion

$$\frac{8.5}{\alpha} = \frac{439}{435}, \tag{1}$$

much as Halley had suggested.

In an identical manner, Short compares observations from fourteen other locations to those taken at the Cape, and concludes that "by taking a mean of the results of these fifteen observations, the parallax of the Sun, on the day of the transit, comes out $= 8.47''$, and by rejecting the 2d, the 8th, the 12th, and the 14th results, which differ the most from the rest, the Sun's parallax, on the day of the transit, by the mean of the eleven remaining ones is $= 8.52''$." He then uses this value to compute the mean equatorial horizontal solar parallax, which can be accomplished as follows. First, recall from FIGURE 1 that the radius of the Earth, which is of course constant, is $A \sin\alpha$. If A_t is the Earth-Sun distance (in au) on the day of the transit and α_m is the mean equatorial horizontal solar parallax (which corresponds to an Earth-Sun distance of $A = 1$ au), then

$$A_t \sin 8.52'' = \sin\alpha_m.$$

Clearly, Short knew that $A_t \approx 1.015$ au, allowing him to compute α_m, for he writes "The parallax of the Sun being thus found, by the observations of the internal contact at the egress, $= 8.52''$ on the day of the transit, the mean (equatorial) horizontal parallax of the Sun is $= 8.65''$." Thus the solar parallax computation is complete. The length of the astronomical unit in miles is now simply $r_e/\sin 8.65''$, where r_e is the radius of the Earth in miles.

But there is a gaping hole in our understanding of Short's method. To complete our understanding, we must develop a way to determine the $6'\,8''$ and $1'\,11''$ time values noted above (and similar values for other observer locations), which arise from the hypothesis of an $8.5''$ solar parallax on the day of the transit. We shall approach the problem in a manner that is undoubtedly different from what Short used in 1761, preferring to use the tools of vector and matrix algebra that are so familiar to us.

Our first task is to develop two coordinate systems and relate them to one another. FIGURE 4 shows the *geocentric equatorial* coordinate system $x'y'z'$ whose origin is at the center of the Earth. The $x'y'$ plane contains the Earth's equator, and the z' axis passes through the north pole. The positive x' axis is oriented so that it passes through the center of the Sun on the first day of spring, and is fixed in space; that is, the Earth's daily motion and annual motion do not change the orientation, but only the location of the origin. Thus the angle θ in FIGURE 4 changes continuously as the Earth rotates.

Now consider an observer at longitude λ and latitude β, measured with the convention that $-180° < \lambda \leq 180°$ and $-90° \leq \beta \leq 90°$, with $\lambda > 0$ east of Greenwich and $\beta > 0$ north of the equator. If θ represents the angular position of Greenwich with respect to the x' axis at a particular instant, then an observer at longitude λ and latitude β will have $x'y'z'$ coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} r_e \cos\beta \cos(\theta + \lambda) \\ r_e \cos\beta \sin(\theta + \lambda) \\ r_e \sin\beta \end{pmatrix}. \tag{2}$$

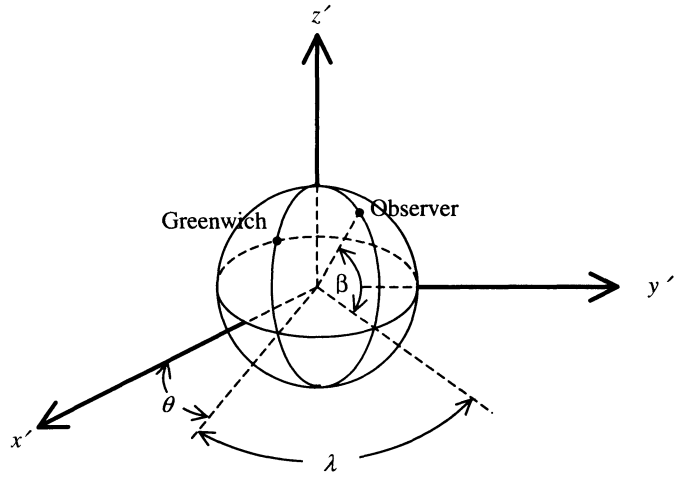


Figure 4 Geocentric equatorial coordinates

This is just the usual spherical-to-rectangular coordinate conversion, with the observation that latitudes are measured up from the equator rather than down from the north pole, as is standard in calculus texts.

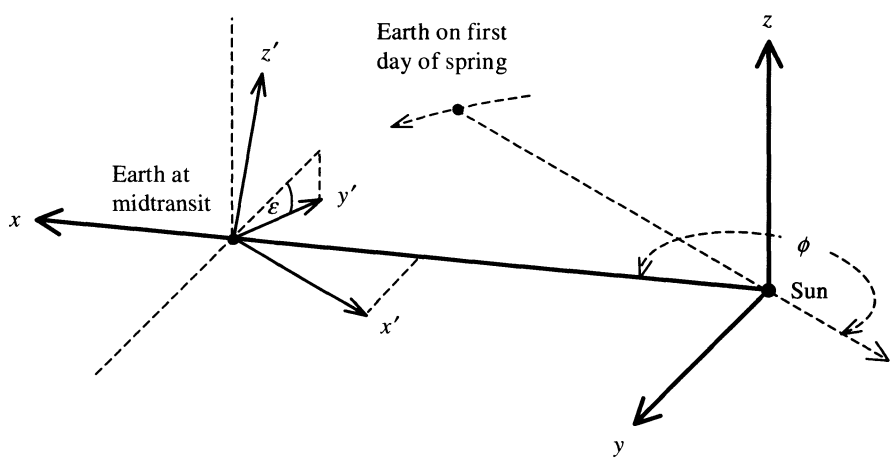


Figure 5 Geocentric equatorial and heliocentric Venus transit coordinates

FIGURE 5 shows the $x'y'z'$ coordinate system along with a second system xyz that we shall call the *heliocentric Venus transit* coordinate system. The xyz coordinate system is chosen with the origin at the center of the Sun, the xy plane containing the Earth's orbit, and the positive x axis directed through the center of the Earth *at midtransit as viewed from the center of the Earth.* The xy plane is known as the *ecliptic* plane.

Converting from one coordinate system to the other is accomplished as follows: first, a rotation about the x' axis through the angle ϵ (the tilt of the Earth's axis with respect to the ecliptic plane) makes the $x'y'$ plane coincide with the xy plane; next, a rotation about the z' axis (now pointing in the same direction as the z axis) through an angle of ϕ makes the x' axis coincide with the x axis; finally, the $x'y'z'$ coordinate

system is translated one unit in the positive x direction. The angle ϕ represents the Earth's position with respect to the Sun at midtransit June 6, 1761, and placing the center of the Earth exactly one unit from the center of the Sun at midtransit is simply a computational convenience. (This distance unit, which we shall use to measure *all* distances in the following discussion, is approximately one astronomical unit, but not exactly so because the Earth is not at its mean distance from the Sun on June 6.) The rotations and translation are accomplished via

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

The Earth's daily motion and annual motion must be accounted for as time elapses from $t = 0$, the moment of midtransit (as viewed from the center of the Earth). For the former, we increase θ in (2) according to $\theta = \theta_0 + 15t$ (degrees), where θ_0 is the position of Greenwich (with respect to the positive x' axis) at $t = 0$ and t is measured in hours. Of course the 15 arises from the fact that the Earth rotates 15° per hour. We shall approximate the Earth's annual motion for the short duration of the transit by assuming that it takes place entirely in the positive y direction. Denoting the Earth's angular velocity at the time of the transit by ω_e, the displacement at time t due to the Earth's annual motion is approximated by $[0, \omega_e t, 0]^T$. By adding this displacement to the right side of (3) and by using (2) to determine $[x', y', z']^T$, we can determine the xyz coordinates at time t of an observer at longitude λ and latitude β.

The essence of our method is to derive vector equations based on the simple observation that the center of the Earth, the center of Venus, and the center of Venus's image on the Sun (as viewed from the center of the Earth) must be collinear. We shall then repeat the computation, replacing the center of the Earth with the position of an observer on the surface of the Earth. FIGURE 6 shows the track of Venus's image across the disk of the Sun, as viewed from the center of the Earth. At $t = 0$ (midtransit), the center of Venus's image on the Sun (in xyz coordinates) is at $\mathbf{I}_0 = [0, d\cos u, d\sin u]^T$, where d and u will be computed from observations. The center of the Earth is at $\mathbf{E}_0 = [1, 0, 0]^T$. Assuming that, for the short duration of the transit, the motion of Venus takes place in the plane $x = x_v$, simple vector addition shows that the position of the center of Venus at time $t = 0$ is $\mathbf{V}_0 = \mathbf{I}_0 + x_v(\mathbf{E}_0 - \mathbf{I}_0)$.

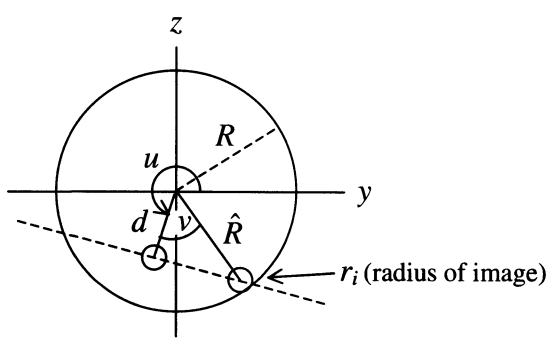


Figure 6 Track of Venus's image on the disk of the Sun

FIGURE 6 shows that at time $t = T$ (the moment of internal contact at egress as viewed from the center of the Earth), the center of Venus's image is at $\mathbf{I}_T = [0, \hat{R}\cos(u + v), \hat{R}\sin(u + v)]^T$. Meanwhile, the center of the Earth has moved to

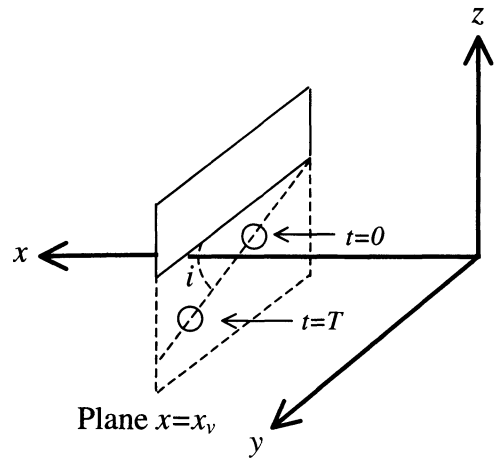


Figure 7 Motion of Venus in the plane $x = x_V$

$\mathbf{E}_T = [1, \omega_e T, 0]^T$. In FIGURE 7, which shows the motion of Venus from $t = 0$ to $t = T$, the angle i is the inclination of Venus's orbit to the xy plane. If ω_v is the angular velocity of Venus, then the distance the planet travels from $t = 0$ to $t = T$ is approximated by $x_v \omega_v T$, so that its location at time T is

$$\mathbf{V}_T = \mathbf{V}_0 + x_v \omega_v T \begin{pmatrix} 0 \\ \cos i \\ -\sin i \end{pmatrix}. \tag{4}$$

The vector $\mathbf{V}_T - \mathbf{I}_T$ must be a constant multiple of the vector $\mathbf{E}_T - \mathbf{I}_T$ in order for the center of the Earth, the center of Venus, and the center of Venus's image on the Sun to be collinear, and the first coordinates tell us that the constant is x_v. By equating the second coordinates in the vector equation $\mathbf{V}_T - \mathbf{I}_T = x_v(\mathbf{E}_T - \mathbf{I}_T)$, expanding $\cos(u + v)$ and simplifying, we are able to obtain

$$\sin u = \frac{x_v(\omega_e - \omega_v \cos i)}{(1 - x_v)\sqrt{\hat{R}^2 - d^2}} T. \tag{5}$$

Likewise, equating the third coordinates, expanding $\sin(u + v)$ and simplifying, we get

$$\cos u = -\frac{x_v \omega_v \sin i}{(1 - x_v)\sqrt{\hat{R}^2 - d^2}} T. \tag{6}$$

The identity $\sin^2 u + \cos^2 u = 1$ allows us to obtain

$$T = \frac{(1 - x_v)}{x_v} \sqrt{\frac{\hat{R}^2 - d^2}{\omega_v^2 + \omega_e^2 - 2\omega_v \omega_e \cos i}}. \tag{7}$$

Once T is known, equations (5) and (6) allow us to determine $\sin u$ and $\cos u$.

Next we repeat the computation for a viewer on the Earth's surface at longitude λ and latitude β. Let $t = T_0$ be the time at which our observer sees the internal contact at egress, where once again $t = 0$ refers to the moment of midtransit as seen from the center of the Earth. Let $\mathbf{I}_{T_0} = [0, y_c, z_c]^T$ designate the center of the image as seen by

the observer at time $t = T_0$, and note that $y_c^2 + z_c^2 = \hat{R}^2$. As noted in the paragraph following (3), the observer's position at $t = T_0$ is

$$\mathbf{O}_{T_0} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_e T_0 \\ 0 \end{pmatrix},$$

where the vector $[x, y, z]^T$ is exactly as given in (3). By changing T to T_0 in (4), we obtain the position of the center of Venus at time $t = T_0$, that is

$$\mathbf{V}_{T_0} = \mathbf{V}_0 + x_v \omega_v T_0 \begin{pmatrix} 0 \\ \cos i \\ -\sin i \end{pmatrix}.$$

As before, the vector $\mathbf{V}_{T_0} - \mathbf{I}_{T_0}$ is a constant multiple of the vector $\mathbf{O}_{T_0} - \mathbf{I}_{T_0}$, and the first coordinates force the constant to be x_v/x. Equating the second coordinates gives us $y_c = M + NT_0$, where

$$M = \frac{(1 - x_v)xd\cos u - x_v y}{x - x_v} \quad \text{and} \quad N = \frac{x x_v \omega_v \cos i - x_v \omega_e}{x - x_v},$$

and equating the third coordinates gives us $z_c = P - QT_0$, where

$$P = \frac{(1 - x_v)xd\sin u - x_v z}{x - x_v} \quad \text{and} \quad Q = \frac{x x_v \omega_v \sin i}{x - x_v}.$$

Then $y_c^2 + z_c^2 = \hat{R}^2$ becomes $(M + NT_0)^2 + (P - QT_0)^2 = \hat{R}^2$, which can easily be solved for T_0. (The + sign in the quadratic formula gives the correct root.)

Our observer at longitude λ and latitude β should see the internal contact at egress $T_0 - T$ hours later than a hypothetical observer at the center of the Earth, assuming the difference is positive, and $|T_0 - T|$ hours earlier if the difference is negative. To illustrate, we restate the values given previously: for an observer at the Cape of Good Hope, $T_0 - T = 0.10222$ hours, or $6' 8''$ later, whereas for an observer at Greenwich, $T_0 - T = -0.01972$ hours, or $1' 11''$ earlier.

We now address the problem of determining values for the many parameters that have been introduced into our work. There are three sources for values of these parameters: First, Short specifically lists a few of the values that he uses; second, and most importantly, many of the values can be computed from Short's values using the underlying hypothesis that the solar parallax is $8.5''$ on the day of the transit; third, there are a few values related to the Earth's daily and annual motions that Short undoubtedly knew, but did not specify. For these last-mentioned values, we resort to modern sources that readily supply the necessary information. In all cases, we shall use Short's values for the latitudes and longitudes of the observers as given in Short's first paper, for accurate determination of the longitude was a significant problem in 1761, and the use of modern values would seriously affect the results. In using Short's longitude values, one must be careful to note that not all are measured with respect to Greenwich, a standard that evolved sometime after 1761.

Let us reiterate that our coordinate system is chosen so that the centers of the Earth and Sun are exactly one unit apart at midtransit, as seen from the center of the Earth. All distances in the following work will be measured in terms of this unit. Under the $8.5''$ hypothesis, the radius of the Earth is therefore $r_e = \sin 8.5''$. Short gives the difference in the parallaxes of Venus and the Sun as $21.35''$ on the day of the transit, so we may take $29.85''$ as the parallax of Venus. Therefore the Venus-to-Earth distance is given by $r_e/\sin 29.85'' = \sin 8.5''/\sin 29.85''$, so that $x_v = 1 - \sin 8.5''/\sin 29.85''$.

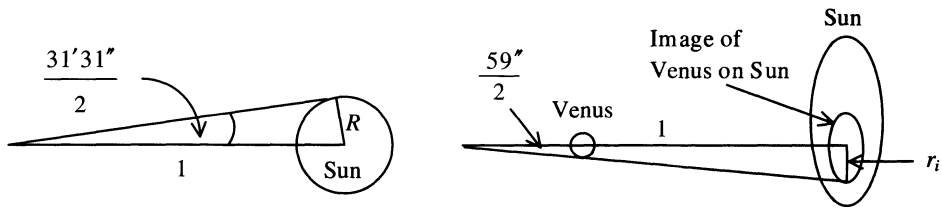


Figure 8 The radii of the Sun and the image of Venus from their angular diameters

Next, Short's value of 31′ 31″ for the angular diameter of the Sun on the day of the observation gives us a value for the solar radius $R = \sin(31'31''/2)$; likewise his value of 59″ for the angular diameter of Venus gives us the value $r_i = \tan(59''/2)$ (see FIGURES 6 and 8). Thus $\hat{R} = R - r_i$ is known. Short also gives the minimum angular separation of the centers of Venus and the Sun as 9′ 32″, as seen from the center of the Earth. This figure, based on the actual transit observation, gives us $d = \tan(9'32'')$. And last, Short gives us two values regarding the motion of Venus, the first of which is $\omega_v = 3'\,59.8''$ of arc per hour. The second is i, the inclination of Venus's orbit with respect to the plane of the Earth's orbit. The value published in Short's paper [**10**] is $i = 8°\,30'\,10''$, *a value that is surely the result of a typesetting error.* For this value produces nonsensical results, whereas the value $i = 3°\,30'\,10''$ not only agrees well with the modern value [**9**] but produces results that match Short's quite well. It is inconceivable that the best data available in 1761 had a 5° error in the inclination.

For the parameters that Short omits from his paper, modern references by Montenbruck and Pfleger [**8**] and Roy [**9**] provide us with the appropriate 1761 values. For the Earth's daily and annual motions, we have used $\epsilon = 23.47°$, $\phi = 256°$, $\theta_0 = -25°$, and $\omega_e = 2'\,25.0''$ per hour. To obtain θ from θ_0, we have assumed the constant t value of 3 hours, which approximates the semi-transit time and which therefore allows us to determine the observer's xyz coordinates at the moment of internal contact at egress. Computational experience suggests that the calculations are quite sensitive to changes in ω_e, but much less so for ϵ, ϕ, and θ.

The lack of certainty as to the exact values James Short used for i, ϵ, ϕ, θ_0, and ω_e, the sensitivity of the computations to ω_e, and our rather different method of computing the $T_0 - T$ values make it impossible to match Short's values exactly. But the results are consistently close, differing from Short's by roughly 1%. Thus the $T_0 - T$ value for the Cape of Good Hope, computed by the above method, is 6′ 12″, compared to Short's value of 6′ 8″. The table below lists data for four other locations, the last being a location in present day Finland. The second column shows our computed $T_0 - T$ value followed by Short's value in parentheses. The third column is the difference between the Cape and the given location, again with Short's value in parentheses. The fourth column shows the difference in time of internal contact at egress between the given location and the Cape, as actually reported by the observers. The last column shows the resulting solar parallax on the day of the transit computed as in (1), with Short's value in parentheses.

Location	$T_0 - T$	Diff. from Cape	Observed	Parallax
Greenwich	−1′ 12″ (−1′ 11″)	7′ 24″ (7′ 19″)	7′ 15″	8.33″ (8.42″)
Rome	−0′ 14″ (−0′ 13″)	6′ 26″ (6′ 21″)	6′ 26″	8.50″ (8.61″)
Stockholm	−2′ 20″ (−2′ 18″)	8′ 32″ (8′ 26″)	8′ 25″	8.38″ (8.48″)
Cajaneburg	−3′ 1″ (−2′ 59″)	9′ 13″ (9′ 7″)	8′ 56″	8.24″ (8.33″)

For further comparison, we note that Short's computed values for the solar parallax on the day of the transit fall between 8.07″ and 8.86″, based on data from the fifteen sites studied in his first paper [**10**].

Short's second method

Short's second method for computing the solar parallax is really just a small variation on the first method, and so it will be very easy for us to describe. This method compares the observed *duration* of the transit, that is, the time between internal contact at ingress and internal contact at egress, to the theoretical duration of the transit computed from the 8.5″ hypothesis. In the notation of the previous section, the theoretical duration for an observer at the center of the Earth is simply $2T$, which can readily be computed from (7). By this method, the duration is $5^h\,57'\,59''$, whereas Short gives the value $5^h\,58'\,1''$. For an observer on the surface of the Earth, one can compute a theoretical duration from the 8.5″ hypothesis as follows. First, compute T_0 (the time between midtransit and internal contact at egress) just as before. Next, by evaluating $\theta = \theta_0 + 15t$ at $t = -3$ instead of $t = 3$, and by using the $-$ sign in the quadratic formula used to determine T_0, we obtain the time *before* midtransit at which the internal contact at ingress should occur. Subtracting this (negative) value from the original T_0 value gives us the theoretical duration of the transit for our observer. We can then use a proportion much like (1) to reconcile the actual observed duration with this theoretical duration.

For example, Short's theoretical duration for Tobolsk is $5^h\,48'\,58''$, which differs from his center-of-the-Earth duration by $9'\,3''$ or 543 seconds. But the observed duration at Tobolsk was $5^h\,48'\,50''$, which differs from his center-of-the-Earth duration by $9'\,11''$ or 551 seconds. Then

$$\frac{8.5}{\alpha} = \frac{\text{center-of-Earth} - \text{theoretical}}{\text{center-of-Earth} - \text{observed}} = \frac{543}{551},$$

yielding a solar parallax of 8.63″ on the day of the transit. To illustrate further, our theoretical duration for Cajaneburg is $5^h\,49'\,54''$, Short's is $5^h\,49'\,56''$, and the observed duration was $5^h\,49'\,54''$. For Stockholm, our theoretical value is $5^h\,50'\,27''$, Short's is $5^h\,50'\,27''$, and the two reported observations are $5^h\,50'\,45''$ and $5^h\,50'\,42''$.

Short uses this second method to compute the solar parallax on the day of the transit using data from sixteen different observers. He thus obtains sixteen values ranging from 8.03″ to 8.98″, with a mean of 8.48″. In a manner that modern statisticians can only envy for its simplicity, he concludes that "if we reject the observations of number 7th, 8th, 9th, 10th, 12th, 13th, and 14th, which differ the most from the rest, the mean of the nine remaining ones gives the Sun's parallax = 8.55", agreeing, to a surprising exactness, with that found by the observations of the internal contact at the egress." If half of the data doesn't support your conclusion, just use the other half!

The transit of 1769 and conclusions

The reader may have sensed by now that the transit of 1761 did not produce the definitive result that Halley had predicted in his 1716 paper. Despite the cleverness of the method and the extraordinary efforts that had gone into making the observations, the conclusions that were drawn from the data still varied widely, with much of the uncertainty due to the lack of accurate longitude data [**15**]. But the experience gained in 1761 only served to whet the appetites and improve the skills of those who would

follow in 1769, when an equally vast international effort was undertaken to observe the second Venus transit of the decade. Captain James Cook was hired to transport observers to the South Pacific, and our friend Chappe (of Tobolsk in Siberia) observed the transit from Baja California, where he died soon thereafter. The luckless Le Gentil, who missed the 1761 transit as a result of the war, waited eight years in the Indian Ocean area for the 1769 transit, only to be defeated by cloudy weather. And how does Euler fit into the story? He certainly did not witness the 1769 transit, for by then he was totally blind, but this did not stop him from writing about it [**2**]. Other than these few items of trivia, we shall not go into the 1769 transit in any detail, for the mathematics did not change significantly from the work already described.

The range of solar parallax values derived from the 1769 transit, and thus the length of the astronomical unit, drew ever closer to the values accepted today. We close by providing the details of a comparison that was mentioned in the introduction: a modern radar-based value for the astronomical unit is 92,955,000 miles [**9**]. And based on his analysis of the 1769 transit of Venus, Thomas Hornsby [**5**] wrote in 1771 that "The parallax on the 3d of June being 8.65″, the mean parallax will be found to be = 8.78″; and if the semidiameter of the Earth be supposed = 3985 English miles, the mean distance of the Earth from the Sun will be 93,726,900 English miles."

Eight-tenths of a percent difference. Absolutely remarkable.

Notes on the sources A very large number of papers on the transits of Venus in 1761 and 1769 appeared in the *Philosophical Transactions of the Royal Society*. Thanks to a project known as the Internet Library of Early Journals (ILEJ), many of these [**5, 10, 11**] are available at `www.bodley.ox.ac.uk/ilej/`. (Hint: sometimes the "next page" arrows at this site don't work, but adjusting the page number in the URL does.) Also, Halley's paper [**4**] can be found online, starting at `www.dsellers.demon.co.uk/`.

One can find a number of web sites devoted to the upcoming Venus transit of June 8, 2004. In particular, a web site maintained by the U.S. Naval Observatory (`http://aa.usno.navy.mil/data/docs/Venus2004.pdf`) gives precise information about the times of the transit predicted for various locations all over the world. According to this web site, the entire transit will be visible throughout most of Europe, Asia, and Africa. The very end of the transit will be visible in the eastern U.S. just after sunrise.

REFERENCES

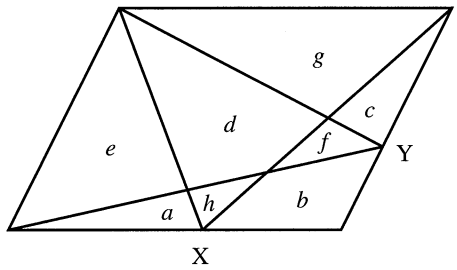
1. A. Acker and C. Jaschek, *Astronomical Methods and Calculations*, John Wiley and Sons, Chichester, 1986.
2. L. Euler, A deduction of the quantity of the sun's parallax from the comparison of the several observations of the late transit of Venus, made in Europe, with those made in George Island in the South Seas, *Philosophical Transactions of the Royal Society* **62** (1772), pp. 69–76.
3. C. Gillispie, editor, *Dictionary of Scientific Biography*, Scribner, New York, 1980.
4. E. Halley, A new method of determining the parallax of the sun, or his distance from the earth, *Philosophical Transactions of the Royal Society, Abridged* **VI** (1809), pp. 243–249.
5. T. Hornsby, The quantity of the sun's parallax, as deduced from the observations of the transit of Venus, on June 3, 1769, *Philosophical Transactions of the Royal Society* **61** (1771), 575–579.
6. W. Ley, *Watchers of the Skies*, Viking Press, New York, 1969 .
7. E. Maor, *June 8, 2004: Venus In Transit*, Princeton University Press, Princeton, New Jersey, 2000.
8. O. Montenbruck and T. Pfleger, *Astronomy on the Personal Computer*, Springer-Verlag, New York, 1991.
9. A. E. Roy, *Orbital Motion*, 3rd ed., Adam Hilger, Ltd., Bristol, UK, 1988.

10. J. Short, The observations of the internal contact of Venus with the sun's limb, in the late transit, made in different places of Europe, compared with the time of the same contact observed at the Cape of Good Hope, and the parallax of the sun from thence determined, *Philosophical Transactions of the Royal Society* **52** (1761), 611–628.
11. J. Short, Second paper concerning the parallax of the sun determined from the observations of the late transit of Venus, in which this subject is treated more at length, and the quantity of the parallax more fully ascertained, *Philosophical Transactions of the Royal Society* **53** (1763), 300–345.
12. L. G. Taff, *Celestial Mechanics: A Computational Guide for the Practitioner*, Wiley, New York, 1985.
13. A. Van Helden, *Measuring the Universe*, University of Chicago Press, Chicago, 1985.
14. A. B. Whatton, *The Transit of Venus Across the Sun... by the Rev. Jeremiah Horrox... to which is prefixed A Memoir of his Life and Labours*, W. Macintosh, London, 1859.
15. H. Woolf, *The Transits of Venus: A Study of Eighteenth Century Science*, Princeton University Press, Princeton, New Jersey, 1959.

Proof Without Words: Equal Areas in a Partition of a Parallelogram

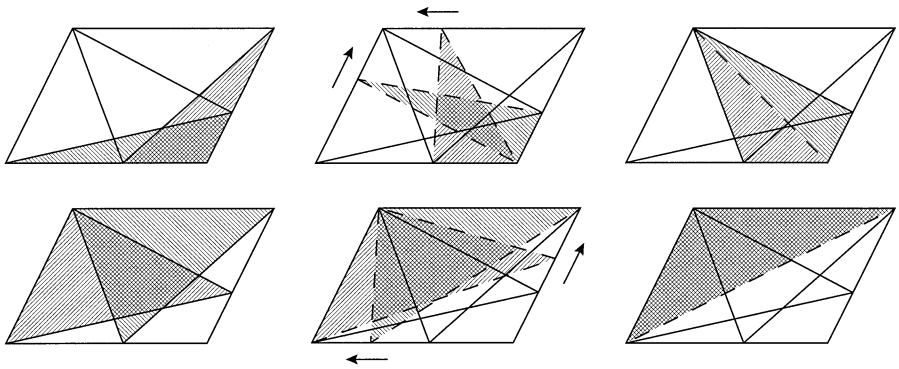
I. $a + b + c = d$

II. $e + f = g + h$



I.

II.



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