FLEXAGONS

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1. Introduction. In 1939, four graduate students at Princeton University (R. Feynman, A. H. Stone, B. Tuckerman, and J. W. Tukey) discovered how to fold a piece of paper into what are now known as *flexagons*—hexagonal gadgets which "flex" under an operation we call pinching to exhibit several faces. Three short notes [3], [4], [6] merely show how to construct two of these paper models. So far as we know there is no other printed literature.

In order to motivate the definitions of Section 4, where abstract flexagons are discussed, we first describe informally how to construct physical models of the simplest abstract flexagons to be called regular flexagons of orders 3, 6, and 9.

2. Regular flexagons of orders 3 and 6. To construct the regular flexagon of order 3, RF_3, take a rectangular strip of paper* about an inch and a half wide and about a foot long, and from one end cut off a 30°, 60° triangle. Next, score

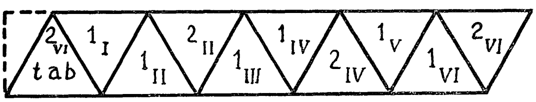

FIG. 1

the strip *very carefully* into ten equilateral triangles; discard any surplus material and label on both sides of the paper as in Figure 1. The first triangle on the left is a tab and is to be glued later to the last triangle on the right. With the strip oriented as in the figure, hold the tab in the left hand and with the right hand fold 2_{II} over on top of 1_{II}, fold 2_{IV} over on top of 1_{IV}, and fold 2_{VI} over on top of 1_{VI}. Now glue tab 2_{VI} onto 2_{VI} and the model is completed.

Hold the flexagon in the position of Figure 2 so that the Roman numerals I to VI, indicating what we shall call *pats*, run clockwise. Each of pats I, III, V contains a single triangle (piece of paper), each of pats II, IV, VI contains two triangles, and the pats are arranged in the form of a hexagon. Mark the vertices of the upper face of 2_{II} with the letters a, b, c clockwise as in Figure 2. Now with pat II to the north, pinch along the east radius (the east half of the east-west diagonal) forcing pats III and IV down. While holding these together push the west end of the west radius down. Actually this causes a folding east and along alternate radii and the flexagon begins to open at the center (now at the top). Releasing the pinching finger and thumb will permit the flexagon to "open" and lie flat again but this time displaying a new face (set of six triangles which will always appear together). The total operation is called a (physical) pinch. While

* Adding machine tape is satisfactory.

2_{II} is at the north, another pinch (east) is impossible since 1_{IV} and 2_{IV} are joined along the east radius. Rotate the whole model $-60°$ and label the vertices of 1_{II}, now north, with a, b, c as before. Pinch east. Rotate $-60°$ and label 1_{I} (north) with a, b, c. Another pinch brings the original face up again so that three distinct faces have appeared. If the flexagon is turned over, three pinches will exhibit three new (mathematical) faces since now the pat numbers run

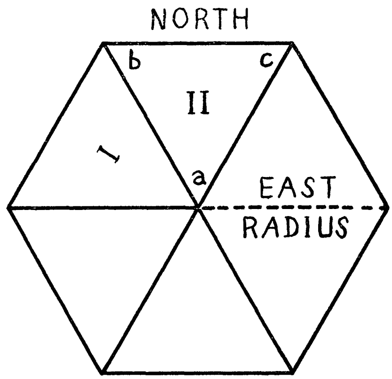

FIG. 2

counterclockwise. Moreover, the letter c is now at the center of the model.

The degree of a pat is the number of triangular pieces of paper in that pat. The order of a flexagon is the sum of the degrees of any two adjacent pats. Here the order is three.

To make the regular flexagon of order 6, RF_6, prepare a strip of nineteen triangles marked as in Figure 3. The construction is accomplished by "winding" the strip up pat by pat. In the folding of the triangles into pats, *the direction of the winding motion of the right hand is that of a wheel rolling on the ground toward you.* In what follows we suppress the Roman subscripts, which are the pat indicators, although they occur in the figure of the strip.

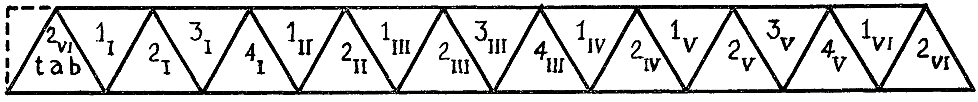

FIG. 3

To wind pat I:

(a) Hold strip as in Figure 3, tab in left hand;
(b) Fold triangle 2 back under triangle 1 by winding motion of right hand;
(c) Fold triangle 4 over on top of triangle 3 by winding motion;
(d) Fold the pair 4, 3 over on top of pair 1, 2 so that, from top down through

the pat, the strip numbers now read 3, 4, 1, 2. This completes the pat; it is of degree 4.

The order 3, 4, 1, 2 of this completed pat is unique; that is, there is no other way to assemble the four triangles in a 4-pat. For the moment we write this order 34,12 with a single comma placed where it tells us exactly what the final folding was in the winding process.

Pat II is a 2-pat and is wound as in pat II of RF_3 above. From top down through the pat the strip numbers are 2, 1 and there is no other way to assemble the two triangles into a 2-pat.

Now wind in order pat III the same as pat I, pat IV the same as pat II, pat V the same as pat I, and pat VI the same as pat II. The windings are therefore a triplication of pats I and II. Glue tab to 2_{VI} and the flexagon is complete. The pinching operation is the same for any flexagon. RF_3 and RF_6 are so special that we omit further discussion, but experimentation with them before proceeding would be helpful.

3. Regular flexagon of order 9. To make a model of the regular flexagon of order 9, RF_9, prepare a strip with a tab and 27 other triangles. Reserving the leftmost triangle for the tab, mark the others, from left to right, 1, 2, 3, 4; 1, 2, 3, 4, 5; · · · , in triplicate. Wind pat I as in RF_6.

To wind pat II:

(a) Hold tab and pat I in left hand;
(b) Put 2 on top of 1;
(c) Put 4 under 3;
(d) Fold pair 3, 4 under pair 2, 1;
(e) Bring 5 over on top of 2 so that, from top down through the pat, the strip numbers read 5,2143. This completes the pat; it is of degree 5.

The position of the comma after 5 tells us that everything to the left of it was folded over 2143 in the last operation. The order 5,2143 is not unique for the 5-pat and this is explained in Section 5.

Triplicate these windings and glue tab. The strip numbers showing on the upper face should be 3's and 5's.

In order to keep track of all of the following pinches, ignore the original strip numbers and place a 1 in the middle of the top triangle of pat II; further, mark this triangle with a, b, c as in RF_3. There is no need of marking the other triangles since the same six pieces of paper will always appear together. This face is now designated as face $1abc$. Record this face on another piece of paper as $1a$ using only the letter at the center. With pat II north, pinch east; label north $2abc$ recording this new face as $2a$, and note that it is impossible to pinch east again. Rotate the flexagon $-60°$, pinch east, label new face (north) $3abc$ and record as $3a$. Pinch, label north $4abc$, record $4a$. Pinch, label north $5abc$, record $5a$. Rotate, pinch; face $3bca$ appears but we record this as $3b$, using only the letter at the center of the model.

Repeat the process: pinch east as many times as possible, then rotate and pinch east, recording new faces as they arise. The succession of faces, from the beginning, is the following, the r indicating that a rotation is to be made before another pinch: $1a$, $2ar$, $3a$, $4a$, $5ar$, $3br$, $4a$, $2c$, $6ar$, $4cr$, $2c$, $3a$, $1a$, $7a$, $8ar$, $1cr$, $7a$, $3c$, $9ar$, $7cr$, $3c$; $1a$.

Remark 1. At any time a pinch following a rotation of $\pm 120°$ will yield the same face as the pinch without such a rotation. That is, pinching along any one of three alternate radii will open the flexagon to the same face. If the flexagon can also be pinched along one of the other set of alternate radii, a second face is exhibited.

Remark 2. In the middle of the sequence of pinches above, the original face $1a$ reappeared. But the next pinch led to the new face $7a$, and not to $2a$, because the flexagon was so oriented that one of the other set of alternate radii lay east. This situation is standard: in every complete sequence of pinches, a given face from which two pinches are possible will appear twice and in the proper orientations to yield the two possible faces. In any flexagon, the faces with no more than a single opening are those and only those where there is but a single paper triangle in alternate pats.

The following definitions apply to all flexagons. A *physical face* is that collection of the six uppermost triangles, one to a pat, regardless of their orientation. Each different orientation of these six triangles, with respect to each other, determines what is called a *mathematical face.*

The above RF_9 has the following properties:

(a) It has 9 physical faces;

(b) It is a Möbius band of 21 half-twists;

(c) It requires 9 rotations and 21 pinches to run through a complete cycle on one side;

(d) It has a total of 30 mathematical faces, 15 on either side. (The pat labels run counterclockwise for each back side mathematical face.)

4. Definition of an abstract flexagon. Take a new flexagon strip, mark with tab, 1, 2, 3, 4, 5 and wind a 5-pat as in RF_9. Now turn the whole strip over noting that the order 52143 has been reversed to read 34125. Mark the next two triangles in the strip 6 and 7, and with these wind a 2-pat which will of course read 76. If this 2-pat is now folded over onto the reversed 5-pat, the numbers 76 are reversed. The total process is that of winding a 7-pat reading, top to bottom, 67,34125. This helps to motivate the following definitions, which in this section are concerned with abstract flexagons. But we do not continue to carry the adjectives "abstract" and "physical" except where confusion might arise.

Let m be a positive integer. For $m=1$, the single permutation of the integer 1 is called a *pat of degree* 1. For $m=r+s>1$, the permutation $A_rA_{r-1}\cdots A_2A_1b_sb_{s-1}\cdots b_2b_1$, where $A_i=a_i+s$, of the integers 1 to m is called a *pat of degree* m if the permutation $a_1a_2\cdots a_r$ of the integers 1 to r and the permutation $b_1b_2\cdots b_s$ of the integers 1 to s are pats of degree r and s, respectively. We define an *abstract flexagon* F to be an ordered pair of pats, $F=(P, Q)$. If the pats are of degree p (a p-pat) and q (a q-pat), then $N=p+q$ is called the *order* of the flexagon F_N.

Two important operations on F_N which preserve the order N are a pinch and a rotation. If the degree of Q is at least two, a *pinch* is that transformation which carries F into F', where $F=(A_rA_{r-1}\cdots A_1b_sb_{s-1}\cdots b_1,\ C_tC_{t-1}\cdots C_1d_ud_{u-1}\cdots d_1)$; $F'=(D_uD_{u-1}\cdots D_1b_1b_2\cdots b_sA_1A_2\cdots A_r,\ c_1c_2\cdots c_t)$; $A_i=a_i+s$, $C_i=c_i+u$, $D_i=d_i+r+s$, and it is easily seen that F' is a flexagon. A *rotation* of a flexagon is the transposition of its pats.

Two flexagons are said to be *equivalent* if one is obtainable from the other by a sequence of pinches or rotations, and this is an equivalence relation.

A flexagon can be represented physically as a triplication of a pair of pats, the arrangement in the paper model being such that the three pairs form a flexible hexagon which, under a pinch, exhibits another face. It can be constructed from a straight piece of paper by folding and gluing. For $N=1+1$, F_2 is the ordinary hexagon.

There is a subclass of flexagons, closed under the operations of pinching and rotating, which constitutes a universal class from which all flexagons arise. A member of this class is called a *regular flexagon* RF_N and is defined as above but in terms of *regular pats* which require $m\not\equiv 0$, mod 3. Further, the degrees p and q of the ordered pair of regular pats must belong to different residue classes, mod 3. Hence p, q are of the form $3k+1$, $3k+2$ and $N=p+q\equiv 0$, mod 3. A model of a regular flexagon can be constructed from a straight piece of paper by folding only.

The next two sections are devoted to regular flexagons.

5. Regular pats. We now consider the number of distinct ways to wind some regular pats of degree m ($\not\equiv 0$, mod 3). Pats of degree $m=1$ and $m=2$ can be constructed in only one way.* A pat of degree $m=4$ is also unique, namely 34,12.

If $m=5$, there are two and only two distinct ways. To make a 5-pat, we add one more triangle to the 4-pat. But clearly this can be done in two ways: first, to the whole 4-pat, *after it has been turned over*, can be added triangle 5, which is now on top. In turning the 4-pat over, we have reversed its sequential order so that the whole 5-pat now reads, top to bottom, 5,2143. Second, a 4-pat could be wound on triangles 2, 3, 4, 5 and its order (3412) would then be 4523. Now *this* can be turned over, becoming 3254, and folded over triangle 1 making the 5-pat read 3254,1. Note that the ordered binary partitions of $5=r+s$, where neither r nor s is congruent to 0, mod 3, are $1+4$ and $4+1$. These must be considered in winding the 5-pat which is necessarily made up of $1+4$ triangles or $4+1$ triangles. There are no other ways to wind a regular 5-pat.

If $m=7$, there are four ways. The permissible ordered binary partitions are $7=2+5$ and $7=5+2$. We take *one* of the 5-pats, say 52143 made from triangles

* We are not concerned here with $m\equiv 0$, mod 3. If you try to wind a 3-pat, for example, you will see that the paper folds back over the tab and the winding cannot proceed. (But see Section 7.)

1, 2, 3, 4, 5, and we take the only 2-pat now to be called 76 because it is made from triangles 6 and 7 in the strip, and combine them. Physically each of these must be turned over to be combined and therefore, sequentially, the 7-pat reads 67,34125. Using, in the same way, the 32541, we get another 7-pat, 67,14523. Or, for the binary partition $7=5+2$, we write the $5=1+4$ sequence, namely 52143, on the triangles 3, 4, 5, 6, 7, thus adding 2 to each member of the sequence which now becomes 74365. Combining this with 21 (reversing each since, physically, they have to be turned over) we arrive at the first sequence corresponding to the partition $7=5+2$, namely, 36745,12. Similarly for the other $7=5+2$, namely, 56347,12. There are no more ordered partitions of 7 and, consequently, no more 7-pats.

We have been doing nothing more than forming regular pats from the definition. Following is a table of regular pats extending through $m=8$. The notation of a pat describes how to wind it. Consider for example the last entry of Table I, and remember the *direction* of winding. Put 5 and 4 together (5 under 4); 7 over 6; 7 (along with 6) over 4; 3 under 2; 8 over 5; 8 (and everything under it) on 2; 6 (and everything under it) on 1.

TABLE I. REGULAR PATS OF DEGREE m

m	Pat	m	Pat
1	1	$8=1+7$	8,5214376
$2=1+1$	2,1		8,2154763
$4=2+2$	34,12		8,2174365
$5=1+4$	5,2143		8,3254176
$=4+1$	3254,1	$=4+4$	6587,2143
$7=2+5$	67,34125	$=7+1$	3265874,1
	67,14523		6325487,1
$=5+2$	36745,12		4365287,1
	56347,12		3285476,1

The *thumbhole* in any pat is that unique place such that each number to the right of it is less than each number to the left. Actually, the comma as it has been used indicates the thumbhole. Since the thumbhole is unique, the comma will now be omitted. The thumbhole separates a pat according to the partitioning used in the winding and is the one place in the physical pat where the thumb can be inserted without encountering a pocket.

Under a pinch east of any flexagon, with pat II north:

(a) Those triangles in I are retained in I but are reversed;

(b) Those triangles above (to the right of) the thumbhole in II are retained in II but are reversed;

(c) Those triangles below (to the left of) the thumbhole in II are slid out of II, without reversal, onto the top of I (reversed).

In the notation of the definition of a flexagon, our example RF_9 becomes (3412,52143). By (a), (b), and (c), the first pinch produces the ordered pair of regular pats (65872143,1) where the structure of each pat is clearly exhibited. (Of course, we mentally relabel the triangles in writing new pat structures.) This "new pat I" is the one 8-pat derived from the partition $8=4+4$ (See Table I). This corresponds to face $2a$. We rotate, pinch, *etc.* The total sequence is as follows:

1a	3412,52143	*2c*	3674512,21		1,85214376
2ar	65872143,1	*6ar*	82154763,1	*1cr*	63254871,1
	1,65872143		1,82154763		1,63254871
3a	32541,3412	*4cr*	32658741,1	*7a*	21,6734125
4a	6714523,21		1,32658741	*3c*	5634712,21
5ar	83254176,1	*2c*	21,3674512	*9ar*	82174365,1
	1,83254176	*3a*	3412,32541		1,82174365
3br	43652871,1	*1a*	52143,3412	*7cr*	32854761,1
	1,43652871	*7a*	6734125,21		1,32854761
4a	21,6714523	*8ar*	85214376,1	*3c*	21,5634712

Each pinch and rotation has produced another flexagon. As a matter of fact, each pat in Table I has been used. This equivalence class of flexagons could have been made originally according to any entry in the above sequence. For example, look at $3a$ 3412,32541; this is an RF_9 wound with the *other* regular 5-pat.

6. Number of regular flexagons. To find the number of regular flexagons of a given order we must first determine the number u_{3k+1} and the number u_{3k+2} of distinct regular pats of degree $3k+1$ and $3k+2$. We have,

$$u_1 = 1, \tag{1}$$

$$u_{3k+1} = u_2u_{3k-1} + u_5u_{3k-4} + \cdots + u_{3k-1}u_2, \tag{2}$$

$$u_{3k+2} = u_1u_{3k+1} + u_4u_{3k-2} + \cdots + u_{3k+1}u_1, \tag{3}$$

where the subscripts describe the ordered partitions. We define the two generating functions $f(x) = \sum_{k=0}^{\infty} u_{3k+1}x^k$ and $g(x) = \sum_{k=0}^{\infty} u_{3k+2}x^k$ and it follows from (1), (2), and (3) that $g(x) = f^2(x)$, $f(x) = 1 + xg^2(x)$, so that

$$f(x) = 1 + xf^4(x), \tag{4}$$

$$g(x) = [1 + xg^2(x)]^2. \tag{5}$$

To compute the coefficients u_{3k+1} and u_{3k+2} we apply Lagrange's inversion formula [8] directly to (4) and after the transformation $h(t) = tg(t^2)$ to (5). The results* are

$$u_{3k+1} = \frac{1}{4k+1}\binom{4k+1}{k}, \qquad u_{3k+2} = \frac{1}{2k+1}\binom{4k+2}{k}.$$

* We are indebted to T. S. Motzkin and Hans Rademacher who, independently, transmitted them to us in correspondence.

The total number U_N of regular flexagons of order N, each being an ordered pair of regular pats, is therefore given by

$$\begin{aligned} U_N = U_{3\lambda} &= 2(u_1u_{3\lambda-1} + u_4u_{3\lambda-4} + \cdots + u_{3\lambda-2}u_2) \\ &= 2\sum_{\gamma=0}^{\lambda-1} \frac{1}{4\gamma+1}\binom{4\gamma+1}{\gamma}\cdot\frac{2}{4(\lambda-\gamma-1)+2}\binom{4(\lambda-\gamma-1)+2}{\lambda-\gamma-1}. \end{aligned}$$

This can be summed by making use of a result of Gould [2], namely: $\sum_{k=0}^{n} A_k(a, b)A_{n-k}(c, b) = A_n(a+c, b)$, where

$$A_m(\alpha, \beta) = \frac{\alpha}{\alpha+\beta m}\binom{\alpha+\beta m}{m}.$$

Therefore,

$$U_{3\lambda} = 2\sum_{\gamma=0}^{\lambda-1} A_\gamma(1, 4)A_{\lambda-\gamma-1}(2, 4) \qquad = \frac{6}{4\lambda-1}\binom{4\lambda-1}{\lambda-1}.$$

But these are divided into equivalence classes, and so in that sense the number of unique regular flexagons is considerably smaller than the number of ways in which they may be constructed. We shall now compute the number of equivalence classes.

Each (physical) triangle must—at some stage of pinching—constitute a pat by itself. For if in any pat P of degree exceeding one, we fix on any particular triangle T, then the flexagon can be held (turned over if necessary) so that T is above the thumbhole in P (north) and a pinch (east) reduces the degree of P which, of course, still contains T. Repetition of this process will reduce the pat P containing T to degree one. (Here, P is used as a generic notation for the new pats containing T; and the turning over of the flexagon causes no loss of generality.)

In each adjacent pair of pats there is a sum of N triangles and each time one of them constitutes a pat by itself, a rotation is necessary to continue the normal course of pinching. Hence, there are at most N rotations in the course of pinching through all the physical faces. Furthermore, for each pair of adjacent pats, each pinch annexes triangles to one of the pats by a half-twist and at the same time removes triangles from the other by means of removing a half-twist from that pat. Since a flexagon of order N is a Möbius band of $3N-6$ half-twists, there will be at most $3N-6$ pinches in running through a class of equivalent flexagons. Therefore, a flexagon runs through at most $4N-6$ stages by means of rotations and pinches. For RF_N it is easy to show that the number of stages is either $4N-6$, (full period), or $(4N-6)/3$, (1/3 period). Period 1/3 occurs when and only when the flexagon is equivalent to $(a_1a_2 \cdots a_m, A_mA_{m-1} \cdots A_1a_ma_{m-1} \cdots a_1)$. For the argument see Section 7.

The number of equivalence classes is therefore $U_N/(4N-6)$ in case the period is full. But in cases where the period is 1/3 this number must be increased by $2u_m/3$. Where $N=3\lambda$, we now take cases.

(a) $\lambda=3k$. Since no regular pat has degree a multiple of three, no flexagon of period 1/3 is possible in this case and the number of equivalence classes is

$$U_N^* = \frac{6}{(12\lambda-6)(4\lambda-1)}\binom{4\lambda-1}{\lambda-1} = \frac{1}{(6k-1)(12k-1)}\binom{12k-1}{3k-1}.$$

(b) $\lambda=3k+1$. Here we must add $2u_{3k+1}/3$.

$$U_N^* = \frac{6}{(12\lambda-6)(4\lambda-1)}\binom{4\lambda-1}{\lambda-1} + \frac{2}{3(3k+1)}\binom{4k}{k}$$
$$= \frac{1}{3(4k+1)(6k+1)}\binom{12k+3}{3k} + \frac{2}{3(3k+1)}\binom{4k}{k}.$$

(c) $\lambda=3k+2$. Similar to (b), we obtain

$$U_N^* = \frac{1}{3(2k+1)(12k+7)}\binom{12k+7}{3k+1} + \frac{4}{3(3k+2)}\binom{4k+1}{k}.$$

7. General flexagons. We now wish to discuss non-regular pats and hence remove the restriction that the order of a flexagon be a multiple of three. Consider the regular pat 3412; if we identify the triangles numbered 3 and 4, we then have a pat of degree three which we may write as $\overline{34}12$, or simply 312. This topological identification of triangles may be executed physically by gluing triangles 3 and 4 together. It is easily seen that such identification may be made on any pair of consecutive integers which are adjacent numbers of the pat. Hence, by identifying 1 and 2 in 3412, we obtain $34\overline{12}$, or simply 231. These are the only possibilities for a general pat of degree three. To illustrate further, we obtain the general pats derivable from the regular 5-pat 52143. These are $5\overline{21}43$ $=4132$, $521\overline{43}=4213$, and $5\overline{21}\overline{43}=312$. All the non-regular pats may be obtained from the regular ones in this manner, and they may be combined without restriction (physically: by gluing when necessary) to form new pats of any degree and flexagons of any order. The definitions of a pat, a flexagon, a pinch, and a rotation are given in Section 4.

Pats of degree m are dependent on the ordered binary partitions of $m=r+s$. For example, two 7-pats are obtained from 4213 and 312 by first taking $a_1a_2a_3a_4$ $=4213$, $b_1b_2b_3=312$ (yielding 6457213) and by next taking $a_1a_2a_3=312$, $b_1b_2b_3b_4$ $=4213$ (yielding 6573124). The pats of degrees 1 through 5 are given by: **1**; **21**; 312, 231; 3241, 2431, **3412**, 4213, 4132; 25341, 24531, **32541**, 42351, 34251, 43512, 35412, 45213, 45132, 51423, 51342, **52143**, 53124, 52314. The regular pats are given in bold face type.

We now obtain the number of non-equivalent flexagons of order N. If v_m is the number of pats of degree m, then $v_m = \sum_{k=1}^{m-1} v_{m-k}v_k$. This recursive convolution may be solved by consideration of the generating function $\phi = \sum_{k=1}^{\infty} v_k x^{k-1}$, which satisfies $x\phi^2 - \phi + 1 = 0$ and for which we impose the condition $v_1 = 1$ as demanded by the definition of a 1-pat. Again, by use of Lagrange's inversion formula, we find

$$v_m = \frac{1}{2m-1}\binom{2m-1}{m}.$$

The number V_N of flexagons of order N is the sum of the number of ways in which ordered pairs of pats may be taken, and hence

$$V_N = v_N = \frac{1}{2N-1}\binom{2N-1}{N}.$$

Notice that $V_N = \{(4N-6)/N\} V_{N-1}$.

As before, each flexagon may be pinched or rotated into $4N-6$ equivalent flexagons except that the flexagon $(a_1a_2 \cdots a_m,\ A_mA_{m-1} \cdots A_1a_ma_{m-1} \cdots a_1)$ has 1/3 period and the flexagon $(a_1a_2 \cdots a_m,\ a_1a_2 \cdots a_m)$ has 1/2 period. We must show that these are the only special cases to worry about in computing V_N^*, the number of non-equivalent flexagons of order N.

First, the two cases cited never coincide. That is, if $N = 6M$, then no flexagon of the form $(a_1a_2 \cdots a_{2M},\ A_{2M}A_{2M-1} \cdots A_1a_{2M}a_{2M-1} \cdots a_1)$ is equivalent to a flexagon of the form $(b_1b_2 \cdots b_{3M},\ b_1b_2 \cdots b_{3M})$. For this to occur, the $2M$-pat $a_1a_2 \cdots a_{2M}$ would have to be a duplication of an M-pat $P = c_1c_2 \cdots c_M$ (*i.e.*, $a_1a_2 \cdots a_{2M} = C_MC_{M-1} \cdots C_1c_Mc_{M-1} \cdots c_1$), and $b_1b_2 \cdots b_{3M}$ would have to be a triplication of the same pat P. Now if we identify the triangles of P as being the same triangle, then the flexagon $(a_1a_2 \cdots a_{2M},\ A_{2M}A_{2M-1} \cdots A_1a_{2M}a_{2M-1} \cdots a_1)$ becomes (21,3412) and $(b_1b_2 \cdots b_{3M},\ b_1b_2 \cdots b_{3M})$ becomes either (312,312) or (231,231). But (21,3412) is a regular flexagon, and hence is not equivalent to a non-regular one.

Second, let F_N be a flexagon of order N. We must establish that F_N is full period, 1/2 period, or 1/3 period. We consider the case in which F_N is a 1-pat and an $(N-1)$-pat, and no generality is lost since every flexagon is equivalent to one of this type. It is clear that if F_N is equivalent to fewer than $4N-6$ flexagons, it is necessarily equivalent to one which is composed of an m-pat P_m, $m>1$, and a k-plication of that same P_m. By identifying the triangles of P_m, we arrive at a derived flexagon which is a 1-pat and a k-pat. If $k=1$, then the derived flexagon is F_2, the hexagon, which has 1/2 period; F_N also has 1/2 period. If $k=2$, the derived flexagon is RF_3, of 1/3 period and F_N has 1/3 period. If $k>2$ and if the derived flexagon were full period, then F_N would also be full period. Since this is contrary to the hypothesis, the derived flexagon is not full period, and we repeat the process. Thus, we arrive at a flexagon which is either 1/2 period or 1/3 period, and F_N has the corresponding periodicity. Since RF_N is never of the form (P, P), a *regular* flexagon cannot be of 1/2 period.

Therefore, in computing V_N^*, we must add to V_N the quantity $\frac{1}{2}v_{N/2}V_2^*$ whenever $N\equiv 0$, mod 2, and $\frac{2}{3}v_{N/3}V_3^*$ whenever $N\equiv 0$, mod 3. But since V_2^* and V_3^* are both 1, we have

$$V_N^* = \frac{1}{4N-6}V_N + \left\{\frac{1}{2}\,v_{N/2}\right\} + \left\{\frac{2}{3}\,v_{N/3}\right\}$$
$$= \frac{1}{N}V_{N-1} + \left\{\frac{1}{2}\,v_{N/2}\right\} + \left\{\frac{2}{3}\,v_{N/3}\right\},$$

where the braces indicate inclusion of that term when and only when applicable.

We now give a table showing some values of the various numbers considered, and it is interesting to note how small the class of regular flexagons is in comparison with the class of all flexagons.

TABLE II

N	u_N	U_N	U_N^*	$V_N(=v_N)$	V_N^*
2	1	0	0	1	1
3	0	2	1	2	1
4	1	0	0	5	1
5	2	0	0	14	1
6	0	6	1	42	4
7	4	0	0	132	6
8	9	0	0	429	19
9	0	30	1	1,430	49
10	22	0	0	4,862	150
11	52	0	0	16,796	442
12	0	182	5	58,786	1,424
13	140	0	0	208,012	4,522
14	340	0	0	742,900	14,924
15	0	1,224	24	2,674,440	49,536
16	969	0	0	9,694,845	167,367
17	2394	0	0	35,357,670	570,285
18	0	8,778	133	129,644,790	1,965,058

8. Remarks. A. There is an essential difference between the class of all flexagons and the subclass of regular ones. The statement in Section 7 that $V_N=v_N$ gives the hint: the general flexagon $(a_1a_2 \cdots a_r,\, b_1b_2 \cdots b_s)$ could be studied and considered as just the pat $A_rA_{r-1} \cdots A_1b_sb_{s-1} \cdots b_1$. The regular flexagon, RF_N, $N\equiv 0$, mod 3, could not be studied by means of studying regular pats since no regular pats have degree $m\equiv 0$, mod 3.

B. Notice that if $P=a_1a_2 \cdots a_m$ is a pat, so also is $P'=a_1'a_2' \cdots a_m'$, where $a_j'+a_{m-j+1}=m+1$. Thus, pats occur in *conjugate pairs* or are self-conjugate (*e.g.*, 3412). To construct P', one may label the triangles in the flexagon strip from right to left instead of from left to right, and then carry out the instructions for winding P. Suppose $F=(P, Q)$ is a model of a flexagon. We have discussed pinching and rotating F, but another obvious operation is that of revolving F 180° about an axis through opposite vertices. This carries F into the *conjugate flexagon* (Q', P').

C. We have constructed a "one-sided" theory of flexagons, and this has led to a simple analysis by convolutions. In the theory of pats, the conjugate pats arose by considering *ordered* partitions. Hence, in computing V_N^*, we have counted twice those *models* arising from ordered binary partitions with unequal components. However, when the components of the partitions are equal, the models have been counted only once. In order to compute the number W_N^* of models of flexagons of order $N=2M$, we write $W_{2M}^*=(V_{2M}^*+v_M)/2$. There is no other corresponding change in Table II.

D. In much the same manner as one labels symmetry operations on the faces of a regular polyhedron to obtain the group of symmetries of that configuration, one can find groups associated with flexagons. For example, one group associated with F_3 is S_3. These usually turn out to be dihedral groups, and finding them is useful and interesting in the teaching of elementary group theory.

E. The number v_m of flexagons of order m adds to the very long list of combinatorial interpretations (ranging from election possibilities and postage stamps to continued fractions) of the recursion formula $v_1=1$; $v_m=\sum_{k=1}^{m-1} v_{m-k}v_k$ which has appeared—with variations—many times in the literature. Some interpretations are mentioned by Becker [1], and a paper by Motzkin [5] gives interpretations and generalizations. The numbers u_N of this paper constitute a variation, and the relevance of continued fractions is given by Touchard [7]. Other references are given in these papers.

Added in proof: An article by Martin Gardner on flexagons appeared in *Scientific American*, December, 1956. While non-mathematical in nature, the article indicates rather complete unpublished work by the inventors, and is an account of the interesting history of the gadgets.

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