Math 12 **Name (Print):** ____________

Fall 2013 - Exam 4: §8&11 - 11/14/13 - Write all responses on separate paper. Show your work for credit.

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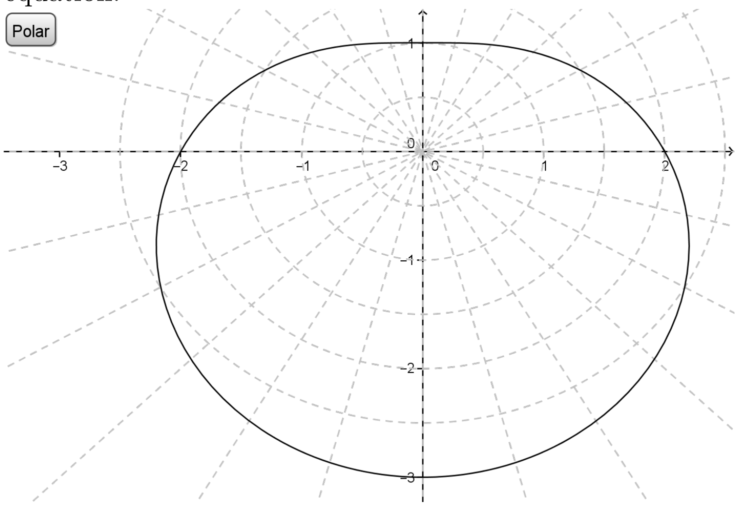
1. Convert the rectangular equation to polar coordinates and solve for r.

   (a) $x^2 + (y-4)^2 = 16$
   Solution: Expanding, we have $x^2 + y^2 - 8y = 0 \Leftrightarrow r^2 = r\sin\theta \Leftrightarrow r = 8\sin\theta$

   (b) $(x^2 + y^2 + y)^2 = 4(x^2 + y^2)$
   $(r^2 + y)^2 = 4r^2 \Leftrightarrow r^4 + 2yr^2 + y^2 - 4r^2 = 0$ Substituting $y = r\sin\theta$ and collecting like terms, $r^4 + 2\sin\theta r^3 + (\sin^2\theta - 4)r^2 = 0$. Now divide through by r^2 (assuming it's not zero) and setting up to complete the square:
   $r^2 + 2\sin\theta r = 4 \Leftrightarrow (r + \sin\theta)^2 = 4$ so $r = -\sin\theta \pm 2$
   Here's a graph of this (note that the origin is also part of the graph since $r = 0$ is a solution to the equation:



2. Convert the polar equation to rectangular coordinates and solve for y.

   (a) $r = \dfrac{1}{\sin\theta + \cos\theta}$
   Solution: Multiply both sides by the denominator and you get $r\sin\theta + r\cos\theta = 1 \Leftrightarrow y = 1 - x$

   (b) $r = \sec\theta(\tan\theta - 1)$
   Solution: Multiply both sides by $\cos\theta$ to get $r\cos\theta = \tan\theta - 1$ then substitute from the Rosetta stone to get $x = \dfrac{y}{x} - 1 \Leftrightarrow y = x^2 + x$

3. Consider the polar function $r = \dfrac{2}{1 - \sin\theta}$

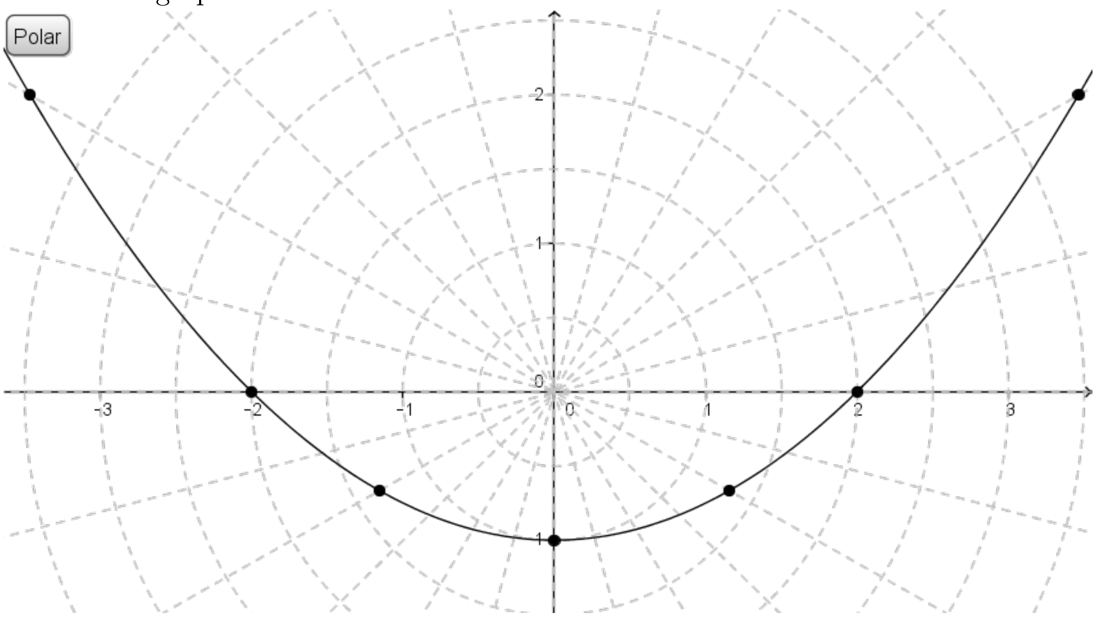
   (a) Test the function for symmetry. What do you find?
   $f(\pi - \theta) = f(\theta)$ so there is y-axis symmetry.

   (b) Write the function as a conic section in standard rectangular form. Multiply both sides by the denominator, solve for r and equate squares to get $r^2 = (y+2)^2 \Leftrightarrow x^2 + y^2 = y^2 + 4y + 4 \Leftrightarrow 4(y+1) = x^2$, a parabola opening upwards from a vertex at $(0, -1)$, with focus at (0,0), directrix along $y = -2$ and with a focal diameter of 4.

(c) Complete the table below for r, x, and y for the given θ

θ	0	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$
r	2	4	4	2	$\frac{4}{3}$	1	$\frac{4}{3}$
x	2	$2\sqrt{3}$	$-2\sqrt{3}$	-2	$-\frac{2\sqrt{3}}{3}$	0	$\frac{2\sqrt{3}}{3}$
y	0	2	2	0	$-\frac{2}{3}$	-1	$-\frac{2}{3}$

(d) construct a graph for the function.

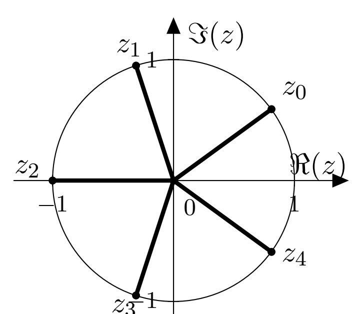


4. Find all solutions to each equation, including the complex solutions. *Hint: first convert the number to polar form and use DeMovire's theorem.*

(a) $x^5 = -1$

Solution: $x^5 = \cos(2k+1)\pi + i\sin(2k+1)\pi \Leftrightarrow x = \cos\left(\frac{(2k+1)\pi}{5}\right) + i\sin\left(\frac{(2k+1)\pi}{5}\right)$ for $k = 0, 1, 2, 3, 4$. Since these expressions are solvable by radicals (see your class notes) we can write these as

$$z_0 = \frac{1+\sqrt{5}}{4} + i\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4},$$
$$z_1 = \frac{1-\sqrt{5}}{4} + i\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4},$$
$$z_2 = -1,$$
$$z_3 = \frac{1-\sqrt{5}}{4} - i\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4},$$
$$z_4 = \frac{1+\sqrt{5}}{4} - i\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4}$$



(b) $x^6 = 8 + 15i$

Solution: Let $\theta_0 = \arctan\left(\frac{15}{8}\right)$. Then $x^6 = 17\left(\cos\left(\theta_0 + 2\pi k\right) + i\sin\left(\theta_0 + 2\pi k\right)\right)$

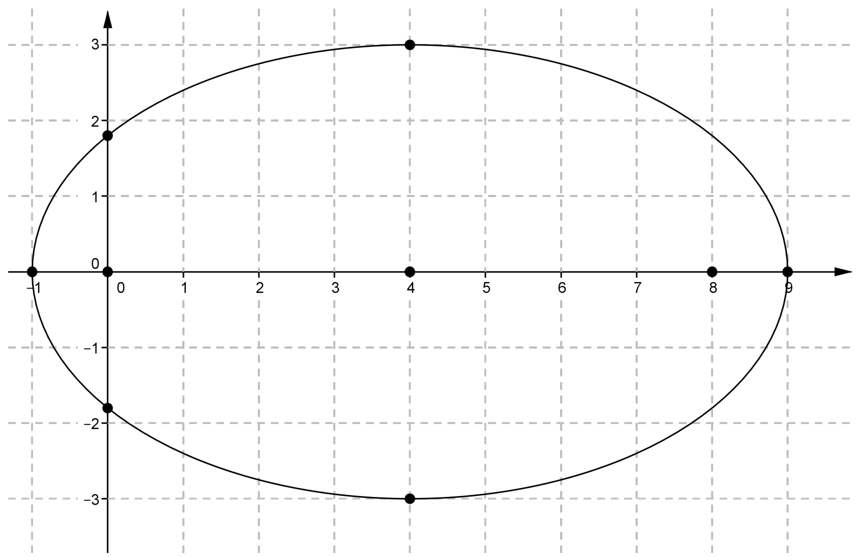
$\Leftrightarrow x = \sqrt[6]{17}\left(\cos\left(\frac{\theta_0 + 2\pi k}{6}\right) + i\sin\left(\frac{\theta_0 + 2\pi k}{6}\right)\right)$ for $k = 0, 1, 2, 3, 4, 5$. I don't think there's any particular insight to be gained by simplifying further.

5. Consider the ellipse described by $\dfrac{(x-4)^2}{25} + \dfrac{y^2}{9} = 1$

(a) Find the center, x-intercepts, y-intercepts and the coordinates of the foci.
Solution: The center is $(4, 0)$ and since $a = 5$ and $b = 3$, $c = \sqrt{25-9} = 4$. Thus the x-intercepts are $(-1, 0)$ and $(9, 0)$. The y-intercepts can be found by setting $x = 0$ and solving for $y = \pm 3\sqrt{1 - \dfrac{16}{25}} = \pm \dfrac{9}{5}$. Finally, the foci are at $(0, 0)$ and $(8, 0)$.

(b) Sketch a graph showing these features.
Solution: It's easier for a graphing device to graph the parametric form, so I'll convert first:

$$\begin{cases} x(t) = 4 + 5\cos(t) \\ y(t) = 3\sin(t) \end{cases}$$



(c) What is the eccentricity, $e = \dfrac{c}{a}$?
$e = \dfrac{c}{a} = \dfrac{4}{5}$

(d) What is the polar form? *Hint: it's in the* $r = \dfrac{ed}{1 - e\cos\theta}$ *form*

Solution: Substituting $c = \dfrac{4}{5}$ and multiplying by $\dfrac{5}{5}$ we have $r = \dfrac{4d}{5 - 4\cos\theta}$ Now $f(0) = 4d$ and $f(\pi) = \dfrac{4d}{9}$ need to match up with the x-intercepts, $(9, 0)$ and $(-1, 0)$, which the will if we set $d = \dfrac{9}{4}$. Thus $r = \dfrac{9}{5 - 4\cos\theta}$

6. Consider the hyperbola describe described by $r = \dfrac{10}{2 - 3\sin\theta}$

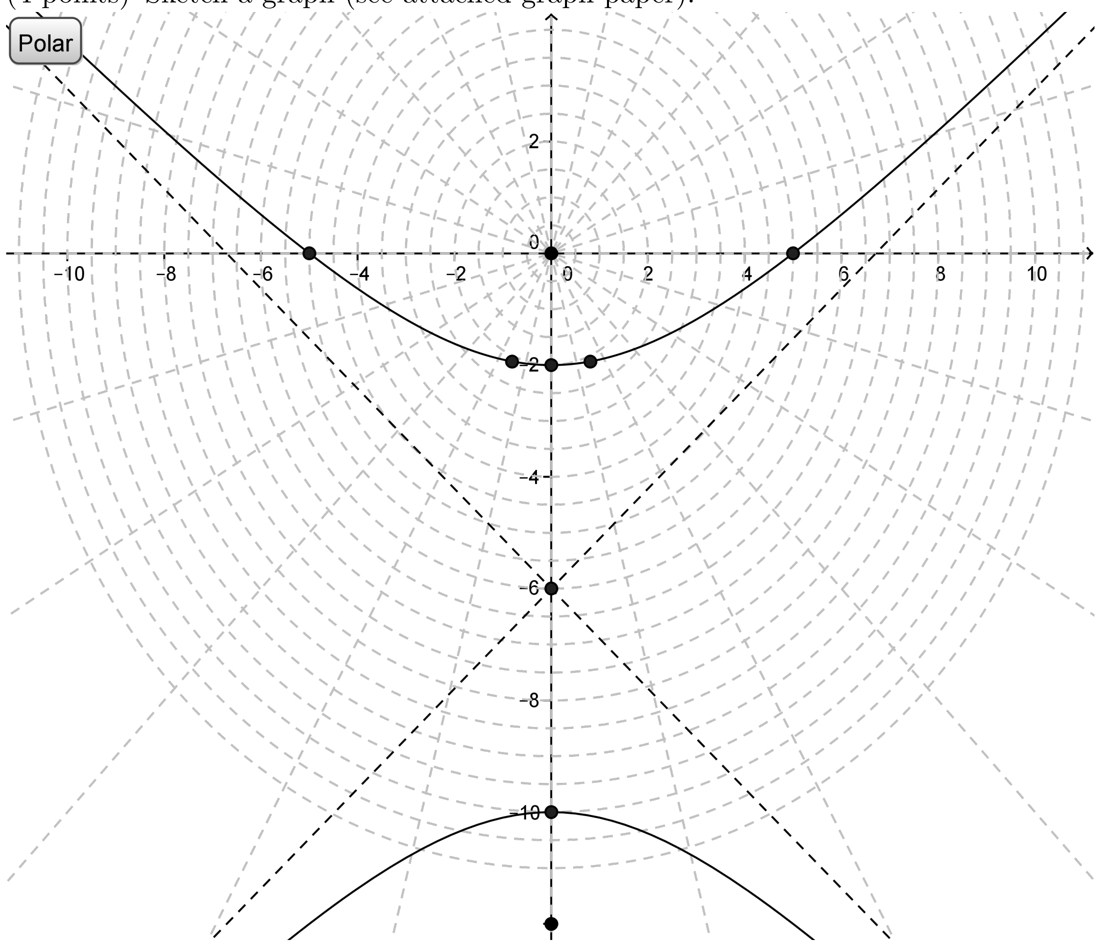
(a) Find the eccentricity.
Solution: $e = \dfrac{3}{2}$.

(b) Complete the table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\arcsin\left(-\frac{12}{13}\right)$	$\pi - \arcsin\left(-\frac{12}{13}\right)$
r	5	-10	5	2	$\frac{65}{31}$	$-\frac{65}{31}$
x	5	0	-5	0	$\frac{25}{31}$	$-\frac{25}{31}$
y	0	-10	0	-2	$-\frac{60}{31}$	$\frac{60}{31}$

(c) Given that the vertices of the hyperbola are the y-intercepts what are the coordinates of the center?
Solution: Halfway between the vertices at $(0,-2)$ and $(0,-10)$ is the center at $(0,-6)$

(d) (4 points) Sketch a graph (see attached graph paper).



(e) (4 points) What is the rectangular form?
Solution: Inspecting the graph, we see that $a = 4, c = 6$ so that $b^2 = 36 - 16 = 20$, whence $\frac{(y+6)^2}{16} - \frac{x^2}{20} = 1$ is the rectangular form.

7. Find parametric equations for each given conic.

(a) $\frac{x^2}{4} + \frac{(y-1)^2}{9} = 1$

$$x = 2\cos(t)$$
$$y = 1 + 3\sin(t)$$

(b) $(x-1)^2 - y^2 = 1$

$$x = 1 + \sec(t)$$
$$y = \tan(t)$$

(c) $4(y-1) = (x-2)^2$

$$x = 2 + 2t$$
$$y = 1 + t^2$$

8. Make a table of values and sketch a graph for the given parametric equations.

$$x = \cos(t) \qquad (1)$$
$$y = \sin^2(t) \qquad (2)$$

t	0	$\pm\frac{\pi}{6}$	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{3}$	$\pm\frac{\pi}{2}$	$\pm\frac{2\pi}{3}$	$\pm\frac{2\pi}{4}$	$\pm\frac{5\pi}{6}$	$\pm\pi$
x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	-1
y	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0

Note that it's easy to eliminate the parameter: $y = 1 - x^2$ but that the graph oscillates only on the part of the parabola where $y \geq 0$