Math 12 Fall 2016 - Exam 4: §8&11 solutions - 12/6/16 -

1. Convert the rectangular equation to polar coordinates and solve for r.

   (a) $(x+2)^2 + y^2 = 4$
   ANS: $(x+2)^2 + y^2 = 4 \Leftrightarrow x^2 + 4x + 4 + y^2 = 4 \Leftrightarrow x^2 + y^2 + 4x = 0 \Leftrightarrow r^2 + 4r\cos\theta = 0 \Leftarrow$ $\boxed{r = -4\cos\theta}$

   (b) $(x-1)^2 - y^2 = 1$ *hint*: use the identity $\cos^2\theta - \sin^2\theta = \cos(2\theta)$
   ANS: $(x-1)^2 - y^2 = 1 \Leftrightarrow x^2 - 2x + 1 - y^2 = 1 \Leftrightarrow r^2\cos^2\theta - 2r\cos\theta - r^2\sin^2\theta = 0 \Leftrightarrow r^2(\cos^2\theta - \sin^2\theta) - 2r\cos\theta = 0 \Leftrightarrow r^2\cos(2\theta) - 2r\cos\theta = 0 \Leftarrow$ $\boxed{r = \dfrac{2\cos\theta}{\cos(2\theta)}}$

2. Convert the polar equation to rectangular coordinates and solve for y.

   (a) $r = 2\cos\theta(1+\tan\theta)$
   ANS: $r = 2\cos\theta(1+\tan\theta) \Leftrightarrow r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2(r\cos\theta + r\sin\theta) \Leftrightarrow x^2 + y^2 = 2(x+y) \Leftrightarrow x^2 - 2x + y^2 - 2y = 0 \Leftrightarrow (x-1)^2 + (y-1)^2 = 2 \Leftrightarrow$ $\boxed{y = 1 \pm \sqrt{2-(x-1)^2}}$

   (b) $r = \dfrac{1}{\sqrt{2}\sin(\theta + \frac{\pi}{4})}$ *hint*: use the addition identity to expand the denominator.
   ANS: $r = \dfrac{1}{\sqrt{2}\sin(\theta + \frac{\pi}{4})} = \dfrac{1}{\sin\theta + \cos\theta} \Leftrightarrow r\cos\theta + r\sin\theta = 1 \Leftrightarrow x + y = 1 \Leftrightarrow$ $\boxed{y = 1 - x}$

3. Consider the polar function $r = 1 + 2\cos\theta$

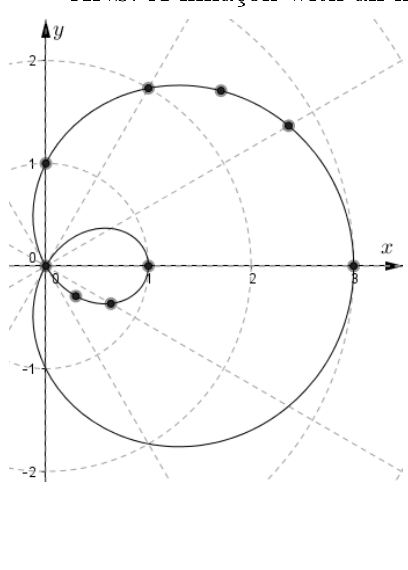
   (a) Test the function for symmetry. What do you find?
   ANS: Since $\cos(-\theta) = \cos\theta$, this curve has $x-$axis symmetry.

   (b) Complete the table below for $r, x,$ and y for the given θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	3	$1+\sqrt{3}$	$1+\sqrt{2}$	2	1	0	$1-\sqrt{2}$	$1-\sqrt{3}$	-1
x	3	$\frac{3+\sqrt{3}}{2}$	$\frac{2+\sqrt{2}}{2}$	1	0	0	$\frac{2-\sqrt{2}}{2}$	$\frac{3-\sqrt{3}}{2}$	1
y	0	$\frac{1+\sqrt{3}}{2}$	$\frac{2+\sqrt{2}}{2}$	$\sqrt{3}$	1	0	$\frac{-2+\sqrt{2}}{2}$	$\frac{1-\sqrt{3}}{2}$	0

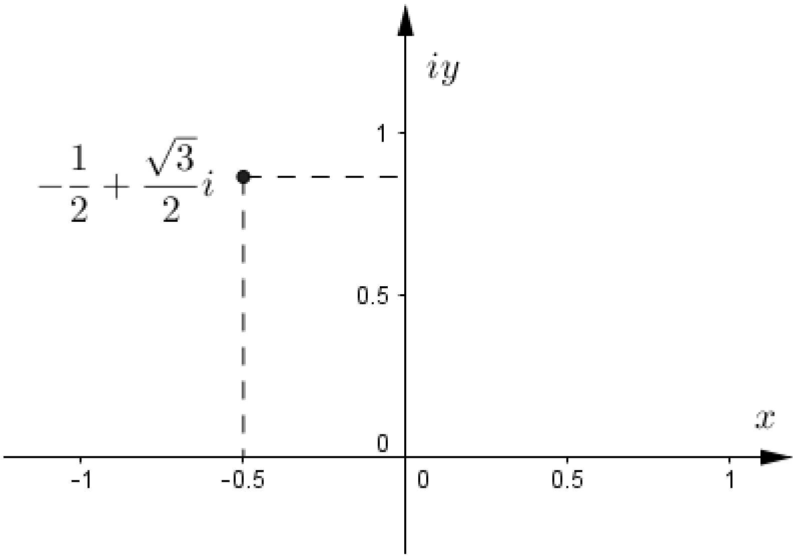
   (c) Construct a complete graph for the function.
   Using the symmetry, we can draw the curve through the tabulated points and then reflect those across the $x-$axis to get the complete graph.

   (d) What kind of curve is this?
   ANS: A limaçon with an inner loop.



4. Let $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$

(a) Plot z in the complex plane.



(b) Write z in polar form.

$z = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

(c) Find z^n for $n = 2, 3$

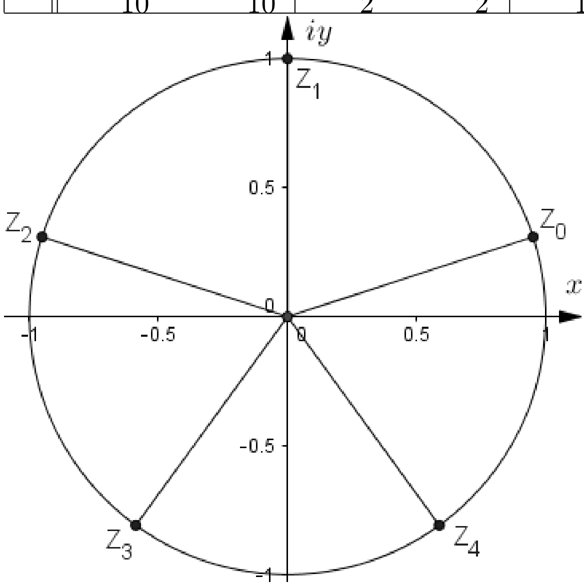
$z^2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$

$z^3 = \cos(2\pi) + i\sin(2\pi) = 1$

5. Find the 5 fifth roots of the imaginary unit, **i** and plot these in the complex plane.

ANS: $i = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)$ so $i^{1/5} = \left(\cos\left(\frac{(1+4k)\pi}{2}\right) + i\sin\left(\frac{(1+4k)\pi}{2}\right)\right)^{1/5}$ for $k = 0, 1, \ldots, 4$

k	0	1	2	3	4
z	$\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$	$\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$	$\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}$	$\cos\frac{13\pi}{10} + i\sin\frac{13\pi}{10}$	$\cos\frac{17\pi}{10} + i\sin\frac{17\pi}{10}$



6. Consider the hyperbola described described by $r = \dfrac{6}{1+2\cos\theta}$

(a) Find the eccentricity.

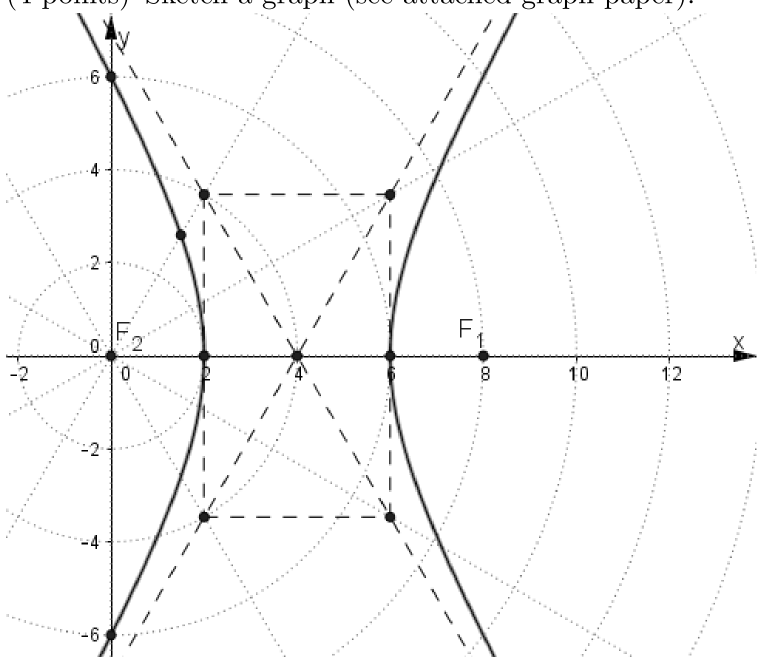
ANS: Since the equation is in the standard form, $r = \dfrac{ed}{1+e\cos\theta}$, we can just read off the value, $\boxed{e=2}$.

(b) Complete the table:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
r	2	6	-6	6	3	NAN
x	2	0	6	0	$3/2$	NAN
y	0	6	0	-6	$3\sqrt{3}/2$	NAN

(c) Given that the vertices of the hyperbola are the x-intercepts what are the coordinates of the center?
ANS: The center is the midpoint of segments between the vertices at $(2,0)$ and $(6,0)$; that is, the center is at $(4,0)$

(d) (4 points) Sketch a graph (see attached graph paper).



(e) (4 points) What is the rectangular form?
Since the distance from the center to the focus is $c=4$ and the distance from the center to a vertex is $a=2$, we have $b=\sqrt{16-4}=2\sqrt{3}\approx 3.4$. The rectangular equation is then $\boxed{\dfrac{(x-4)^2}{16}-\dfrac{y^2}{12}=1}$

7. Use the Pythagorean identity to find parametric equations for each given conic.

(a) $x^2 + \dfrac{(y-3)^2}{4} = 1$
ANS: There are many parameterizations, but perhaps the simplest is
$\boxed{x=\cos(t) \text{ and } y = 3+2\sin(t)}$

(b) $(y-2)^2 - x^2 = 1$
ANS: $\boxed{y = 2+\sec(t) \text{ and } x=\tan(t)}$