Math 12 - Fall 2016 - Exam 3 Solutions (chapter 7)

1. (10 points) Simplify the expression: $\dfrac{2+\tan^2(x)}{\sec^2 x}$

   SOLN: $\dfrac{2+\tan^2(x)}{\sec^2 x} = \dfrac{1+\sec^2(x)}{\sec^2(x)} = \cos^2(x)+1$

2. Prove the identity:

   (a) $(\sin x+\cos x)^2 = 1+\sin(2x)$
   Expand the square on the left: $\sin^2 x + 2\sin x\cos x + \cos^2 x$ and substitute using the Pythagorean identity to get $1+2\sin x\cos x$ Then use the identity $2\sin x\cos x = \sin 2x$ to prove the identity.

   (b) $\dfrac{\sin\theta}{1-\cos\theta} - \cot\theta = \csc\theta$
   $\dfrac{1+\cos\theta}{1+\cos\theta}\dfrac{\sin\theta}{1-\cos\theta} - \cot\theta = \dfrac{\sin\theta+\sin\theta\cos\theta}{1-\cos^2\theta} - \cot\theta = \dfrac{\sin\theta+\sin\theta\cos\theta}{\sin^2\theta} - \cot\theta$
   $= \csc\theta + \cot\theta - \cot\theta = \csc\theta$

3. Prove that $\cos(\alpha+\beta)\cos(\beta) + \sin(\alpha+\beta)\sin(\beta) = \cos\alpha$.
   There are two ways to approach this:
   (1) Try expanding using the addition identities:
   $\cos(\alpha+\beta)\cos(\beta)+\sin(\alpha+\beta)\sin(\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\cos\beta + (\sin\alpha\cos\beta+\sin\beta\cos\alpha)\sin\beta =$
   Now distribute and regroup the terms to use the Pythagorean identity:
   $\cos\alpha(\cos^2\beta+\sin^2\beta) - \sin\alpha\sin\beta\cos\beta + \sin\alpha\sin\beta\cos\beta = \cos\alpha$

   (2) Alternatively, use the product-to-sum identities:
   $\cos(\alpha+\beta)\cos(\beta)+\sin(\alpha+\beta)\sin(\beta) = \frac{1}{2}(\cos(\alpha+2\beta)+\cos\alpha) + \frac{1}{2}(\cos\alpha - \cos(\alpha+2\beta)) = \frac{1}{2}\cos\alpha + \frac{1}{2}\cos\alpha$

4. Find values of C and ϕ so that the given sum is equal to $C]sin(\omega t+\phi)$:

   (a) $\sqrt{2}\cos(5x) + 2\sin(5x) = \sqrt{2+4}\sin(5x+\arctan(\sqrt{2}/2)) = \sqrt{6}\sin(5x+\arctan(\sqrt{2}/2))$

   (b) $\sqrt{3}\sin(\pi x) + 2\cos(\pi x) = \sqrt{3+4}\sin(\pi x + \arctan(2/\sqrt{3})) = \sqrt{7}\sin(\pi x+\arctan(2\sqrt{3}/3))$

5. Show that $\dfrac{\sqrt{2}+\sqrt{6}}{4} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$ by considering that $\sin\left(\dfrac{5\pi}{12}\right) = \sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right) = \sin\left(\dfrac{5\pi/6}{2}\right)$

   SOLN: $\sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right) = \sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{4}\right)+\cos\left(\dfrac{\pi}{6}\right)\sin\left(\dfrac{\pi}{4}\right) = \frac{1}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$
   $\sin\left(\dfrac{5\pi/6}{2}\right) = \sqrt{\dfrac{1+\cos(5\pi/6)}{2}} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$

6. Use the addition identity for $\tan(x)$ to write $\tan(3x)$ in terms of only $\tan(x)$. Simplify. Verify that you formula is correct at $x = \dfrac{\pi}{4}$

   $$\tan(2x+x) = \frac{\tan 2x + \tan x}{1-\tan 2x\tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x}+\tan x}{1-\dfrac{2\tan^2 x}{1-\tan^2 x}}\cdot\frac{1-\tan^2 x}{1-\tan^2 x} = \frac{2\tan x+\tan x-\tan^3 x}{1-\tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$$

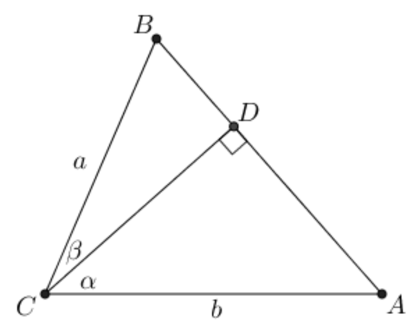
   At $x=\dfrac{\pi}{4}$, $-1 = \tan\left(\dfrac{3\pi}{4}\right) = \dfrac{3-1}{1-3}$ check.

7. (a) Prove that $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
   Let $x = A + B$ and $y = A - B$. Then $2A = x + y$ and $2B = x - y$ so that $\sin x + \sin y = \sin(A + B) + \sin(A - B) = \sin A\cos B + \sin B\cos A + \sin A\cos B - \sin B\cos A = 2\sin A\cos B = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

   (b) Prove that $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
   $\cos(x) + \cos(y) = \cos(A + B) + \cos(A - B) = \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B = 2\cos A\cos B = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

   (c) Prove that $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\frac{x+y}{2}\right)$
   Substitute from the results of parts (a) and (b) and reduce by the common factor of $2\cos\left(\frac{x-y}{2}\right)$ then use the identity, $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

8. Consider the diagram at right which we'll use to prove the addition identity for $\sin(\alpha + \beta)$.



   (a) Express $\overline{AD}$ in terms of b and α. : $\sin\alpha = \frac{\overline{AD}}{b} \Leftrightarrow \overline{AD} = b\sin\alpha$

   (b) Express $\overline{BD}$ in terms of a and β. : $\sin\beta = \frac{\overline{BD}}{a} \Leftrightarrow \overline{BD} = a\sin\beta$

   (c) Express $\overline{CD}$ in two ways:
   - in terms of b and α. : $\overline{CD} = a\cos\beta = b\cos\alpha$
   - in terms of a and β.

   (d) Express the area of $\triangle ABC$ in terms of α, β, a and b:
   Area= $\frac{1}{2}ab\sin(\alpha + \beta)$

   (e) Use the fact that the area of $\triangle ABC$ is the sum of areas of $\triangle CBD$ and $\triangle ACD$ to prove the addition identity.
   $\triangle ABC = \triangle CBD + \triangle ACD \Leftrightarrow \frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}\overline{BD}\cdot\overline{CD} + \frac{1}{2}\overline{AD}\cdot\overline{CD} = \frac{1}{2}ab\sin\beta\cos\alpha + \frac{1}{2}ab\sin\alpha\cos\beta$

9. Solve the equation.

   (a) $(2\cos(3\theta) - 1)(2\cos(3\theta) + 1) = 1$
   $\Leftrightarrow 4\cos^2(3\theta)) - 1 = 1 \Leftrightarrow \cos(3\theta) = \frac{\pm\sqrt{2}}{2} \Leftrightarrow 3\theta = \pm\frac{\pi}{4} + \pi k \Leftrightarrow \boxed{\theta = \frac{\pi}{12} + \frac{k\pi}{3} = \frac{(4k+1)\pi}{12}}\ k \in \mathbb{Z}$

   (b) $\cos(3\theta) - \cos(7\theta) = 0$
   Use the addition identities like so: $\cos(5\theta - 2\theta) - \cos(5\theta + 2\theta)$
   $\cos 5\theta\cos 2\theta + \sin 5\theta\sin 2\theta - (\cos 5\theta\cos 2\theta - \sin 5\theta\sin 2\theta) = 2\sin 5\theta\sin 2\theta = 0$ so either $\sin 5\theta = 0 \Leftrightarrow$
   $\boxed{\theta = \frac{k\pi}{5}}$ or $\boxed{\theta = \frac{k\pi}{2}, k \in \mathbb{Z}}$