Math 12 – Redemption Synthesis Test 2 – spring ’09 Name________________________

Write all responses on separate paper. Show your work for credit.

1. Let $f(x) = \ln\left(2x^3 - x^2 - 4x + 3\right)$.
   a. List the possible rational zeros of $2x^3 - x^2 - 4x + 3$ according to the theorem on rational zeros.

   b. Find all the *x*-intercepts of *f*. Remember that ln(1) = 0 is equivalent to $e^0 = 1$. Show how to do this without a calculator.

   c. List the vertical asymptotes as equations in the form: *x* = constant.

   d. What is the domain of *f* ?

2. Let $f(x) = e^{4x} - 5e^{2x} + 4$
   a. Solve *f*(*x*) = 0.

   b. Solve *f*(*x*) = 4.

3. Find a polynomial with integer coefficients and zeros at $x = \frac{3}{4}$, $x = \frac{\sqrt{2}}{2}$ and $x = \sqrt{2}\,i$.

   Write the polynomial in descending powers form.

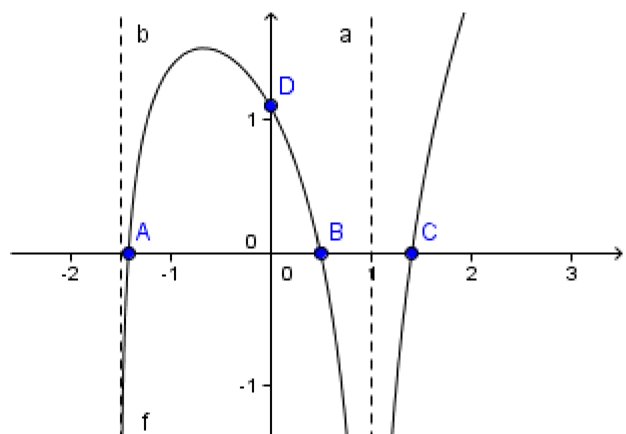
4. Let $p(x) = 8x^3 - 4x^2 - 6x + 3$.
   a. Find the 3 zeros of *p*. Write these in exact, simplified form.

   b. Find at least 4 different solutions to *p*(cos(*t*)) = 0.

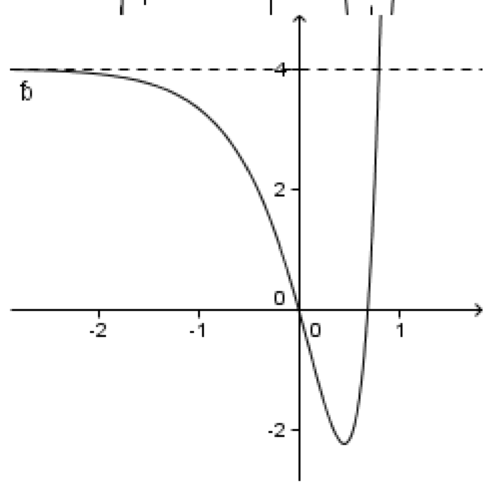
5. Find the vertical asymptotes of $y = \dfrac{1}{2\sin^2(2x) - 3\sin(2x) + 1}$

Math 12 – Redemption Synthesis Test 2 Solutions.

1. Let $f(x) = \ln\left(2x^3 - x^2 - 4x + 3\right)$.
   a. List the possible rational zeros of $2x^3 - x^2 - 4x + 3$ according to the theorem on rational zeros.
   SOLN: $\left\{\pm x \mid x = 1, 3, \frac{1}{2}, \frac{3}{2}\right\}$
   b. Find all the *x*-intercepts of *f*. Remember that ln(1) = 0 is equivalent to $e^0 = 1$. Show how to do this without a calculator.
   SOLN: For the *x*-intercepts, solve $2x^3 - x^2 - 4x + 3 = 1 \Leftrightarrow 2x^3 - x^2 - 4x + 2 = 0$. Since the coefficients are integers, the rational zeros must be in the set $\left\{\pm x \mid x = 1, 2, \frac{1}{2}\right\}$. Since the polynomial changes sign between *x* = 0 and *x* = 1, it makes sense to check *x* = ½ and, indeed, it is a root: $2x^3 - x^2 - 4x + 2 = (2x-1)\left(x^2 - 2\right)$, which you can also find by grouping. So the *x*-intercepts are (½ , 0), $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$.
   c. List the vertical asymptotes as equations in the form: *x* = constant.
   SOLN: The vertical asymptotes are where $2x^3 - x^2 - 4x + 3 = (x-1)^2(2x+3) = 0$. That is, *x* = 1 and $x = -\frac{3}{2}$
   d. What is the domain of *f* ?
   SOLN: The domain is where
   $2x^3 - x^2 - 4x + 3 = (x-1)^2(2x+3) > 0$
   $\Leftrightarrow x \neq 1 \text{ and } 2x + 3 > 0$
   In interval notation, the domain is $\left(-\frac{3}{2}, 1\right) \cup (1, \infty)$.
   As a bonus, it's worth examining the graph: (at right.) Note the *y*-intercept is ln(3), a tad bit bigger than 1, as 3's a little more than *e*.



2. Let $f(x) = e^{4x} - 5e^{2x} + 4$
   a. Solve *f*(*x*) = 0. SOLN:
   $f(x) = e^{4x} - 5e^{2x} + 4 = 0 \Leftrightarrow \left(e^{2x} - 4\right)\left(e^{2x} - 1\right) = 0$
   $\Leftrightarrow \left(e^x - 2\right)\left(e^x + 2\right)\left(e^x - 1\right)\left(e^x + 1\right) = 0$
   $\Leftrightarrow \left(e^x - 2\right)\left(e^x - 1\right) = 0$, so *x* = 0 or *x* = ln2 ~ 0.69.
   b. Solve *f*(*x*) = 4.
   SOLN: $f(x) = e^{4x} - 5e^{2x} + 4 = 4$
   $\Leftrightarrow e^{4x} - 5e^{2x} = e^{2x}\left(e^{2x} - 5\right) = 0$
   $\Leftrightarrow e^{2x} - 5 = 0 \Leftrightarrow x = \ln\sqrt{5}$



3. Find a polynomial with integer coefficients and zeros at $x=\frac{3}{4}$, $x=\frac{\sqrt{2}}{2}$ and $x=\sqrt{2}\,i$.

Write the polynomial in descending powers form.
SOLN: $(4x-3)(2x^2-1)(x^2+2)=(4x-3)(2x^4+3x^2-2)=8x^5-6x^4+12x^3-9x^2-8x+6$

4. Let $p(x)=8x^3-4x^2-6x+3$.

a. Find the 3 zeros of *p*. Write these in exact, simplified form.
SOLN: $p(x)=8x^3-4x^2-6x+3=0$ has integer coefficients so the rational solutions must be contained in the set $\left\{\pm x \mid x=1,3,\frac{1}{2},\frac{3}{2},\frac{1}{4},\frac{3}{4},\frac{1}{8},\frac{3}{8}\right\}$.

in				out	
0	8	-4	-6	3	
1	8	4	-2	1	
3/2	8	8	6	12	← 3/2 is an upper bound on the zeros
-1	8	-12	6	-3	← -1 is an lower bound on the zeros.
-1/2	8	-8	-2	2	and there is a zero between 0 and -1
-3/4	8	-10	3/2	15/8	← the zero between -3/4 and -1 is irrational
1/2	8	0	-6	0	← should've looked there first!

Searching, we find $x = ½$ is a zero, thus, $p(x)=(2x-1)(4x^2-3)$ and the zeros are $\left\{\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right\}$

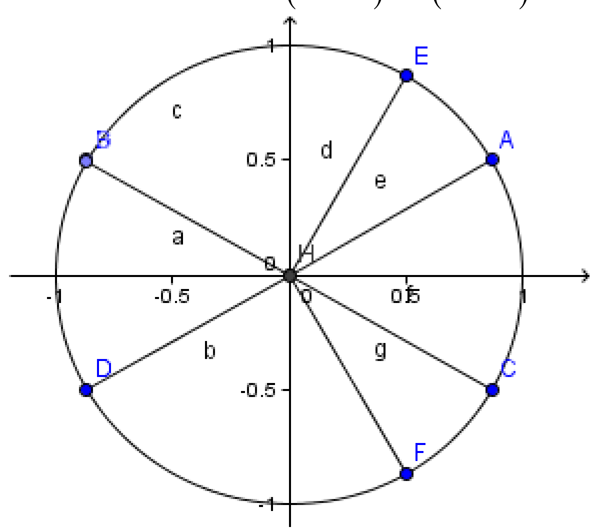
Note that *p* can also be factored by grouping: $p(x)=8x^3-4x^2-6x+3=4x^2(2x-1)-3(2x-1)$

b. Find at least 4 different solutions to *p*(cos(*t*)) = 0.

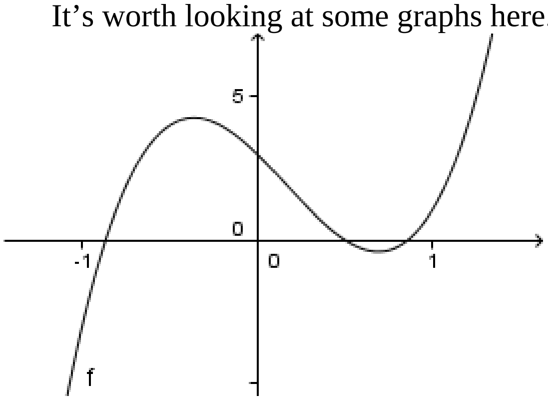
SOLN: From the above, cos(*t*) = ½ or $\cos t=\pm\frac{\sqrt{3}}{2}$.

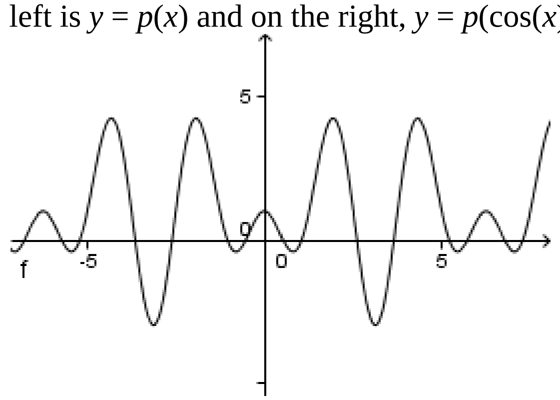
These correspond to the points on the unit circle shown at right. Points E and F correspond to cos(*t*) = ½ and are arrived at by any value of *t* in the form $t=\pm\frac{\pi}{3}+2\pi k, k\in\mathbb{Z}$ and the points A, B, C, D where $\cos t=\pm\frac{\sqrt{3}}{2}$ are described by

$t=\pm\frac{\pi}{6}+k\pi, k\in\mathbb{Z}$



It's worth looking at some graphs here. On the left is *y* = *p*(*x*) and on the right, *y* = *p*(cos(*x*))





5. Find the vertical asymptotes of $y = \dfrac{1}{2\sin^2(2x) - 3\sin(2x) + 1}$

SOLN: The vertical asymptotes are where $2\sin^2(2x) - 3\sin(2x) + 1 = 0$

This factors, $(2\sin(2x) - 1)(\sin(2x) - 1) = 0$. By the zero product principle,

either $2\sin(2x) - 1 = 0 \Leftrightarrow \sin(2x) = \dfrac{1}{2}$ or $\sin(2x) - 1 = 0 \Leftrightarrow \sin(2x) = 1$

This means that $2x = \dfrac{\pi}{2} \pm \dfrac{\pi}{3} + 2\pi k = \dfrac{(4k+1)\pi}{2} \pm \dfrac{\pi}{3} \Leftrightarrow \boxed{x = \dfrac{(4k+1)\pi}{4} \pm \dfrac{\pi}{6}}$

or $2x = \dfrac{\pi}{2} + 2\pi k = \dfrac{(4k+1)\pi}{2} \Leftrightarrow \boxed{x = \dfrac{(4k+1)\pi}{4}}$

The first set contains angles like $\dfrac{\pi}{4} \pm \dfrac{\pi}{6} = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$ and $\dfrac{5\pi}{4} \pm \dfrac{\pi}{6} = \dfrac{13\pi}{12}$ or $\dfrac{17\pi}{12}$

The second set contains angles such as $\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}$, and so on.

Here's a graph of this function with the vertical asymptotes as dashed lines: