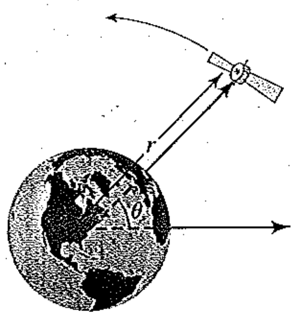
(b) For what angle θ is the satellite closest to the earth? Find the height of the satellite above the earth's surface for this value of θ.



54. **An Unstable Orbit** The orbit described in Exercise 53 is stable because the satellite traverses the same path over and over as θ increases. Suppose that a meteor strikes the satellite and changes its orbit to

$$r = \frac{22500\left(1 - \dfrac{\theta}{40}\right)}{4 - \cos\theta}$$

(a) On the same viewing screen, graph the circle $r = 3960$ and the new orbit equation, with θ increasing from 0 to 3π. Describe the new motion of the satellite.

(b) Use the TRACE feature on your graphing calculator to find the value of θ at the moment the satellite crashes into the earth.

Discovery • Discussion

55. **A Transformation of Polar Graphs** How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

56. **Choosing a Convenient Coordinate System** Compare the polar equation of the circle $r = 2$ with its equation in rectangular coordinates. In which coordinate system is the equation simpler? Do the same for the equation of the four-leaved rose $r = \sin 2\theta$. Which coordinate system would you choose to study these curves?

57. **Choosing a Convenient Coordinate System** Compare the rectangular equation of the line $y = 2$ with its polar equation. In which coordinate system is the equation simpler? Which coordinate system would you choose to study lines?

8.3 Polar Form of Complex Numbers; DeMoivre's Theorem

In this section we represent complex numbers in polar (or trigonometric) form. This enables us to find the nth roots of complex numbers. To describe the polar form of complex numbers, we must first learn to work with complex numbers graphically.

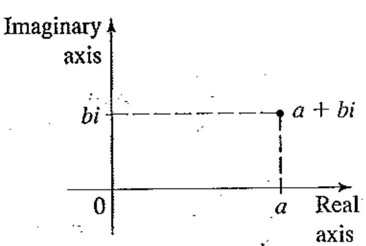


Figure 1

Graphing Complex Numbers

To graph real numbers or sets of real numbers, we have been using the number line, which has just one dimension. Complex numbers, however, have two components: a real part and an imaginary part. This suggests that we need two axes to graph complex numbers: one for the real part and one for the imaginary part. We call these the **real axis** and the **imaginary axis**, respectively. The plane determined by these two axes is called the **complex plane**. To graph the complex number $a + bi$, we plot the ordered pair of numbers (a, b) in this plane, as indicated in Figure 1.

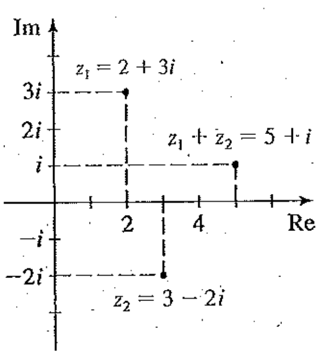


Figure 2

Example 1 Graphing Complex Numbers

Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

Solution We have $z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$. The graph is shown in Figure 2. ∎

Example 2 Graphing Sets of Complex Numbers

Graph each set of complex numbers.

(a) $S = \{a + bi \mid a \geq 0\}$ (b) $T = \{a + bi \mid a < 1, b \geq 0\}$

Solution

(a) S is the set of complex numbers whose real part is nonnegative. The graph is shown in Figure 3(a).

(b) T is the set of complex numbers for which the real part is less than 1 and the imaginary part is nonnegative. The graph is shown in Figure 3(b).

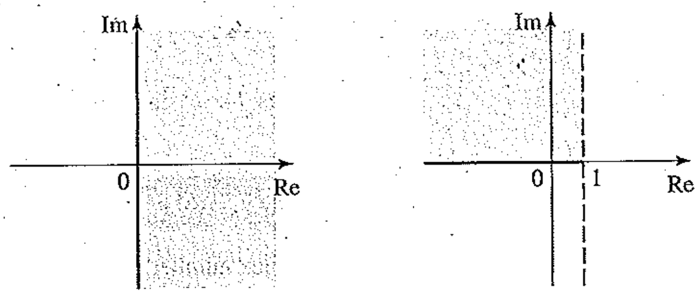


Figure 3 (a) (b)

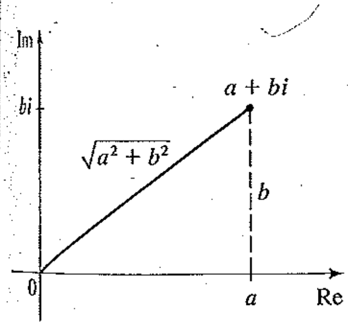


Figure 4

Recall that the absolute value of a real number can be thought of as its distance from the origin on the real number line (see Section 1.1). We define absolute value for complex numbers in a similar fashion. Using the Pythagorean Theorem, we can see from Figure 4 that the distance between $a + bi$ and the origin in the complex plane is $\sqrt{a^2 + b^2}$. This leads to the following definition.

The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$

Example 3 Calculating the Modulus

The plural of *modulus* is *moduli*.

Find the moduli of the complex numbers $3 + 4i$ and $8 - 5i$.

Solution

$$|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|8 - 5i| = \sqrt{8^2 + (-5)^2} = \sqrt{89}$$

Example 4 Absolute Value of Complex Numbers

Graph each set of complex numbers.

(a) $C = \{z \mid |z| = 1\}$ (b) $D = \{z \mid |z| \leq 1\}$

Solution

(a) C is the set of complex numbers whose distance from the origin is 1. Thus, C is a circle of radius 1 with center at the origin, as shown in Figure 5.

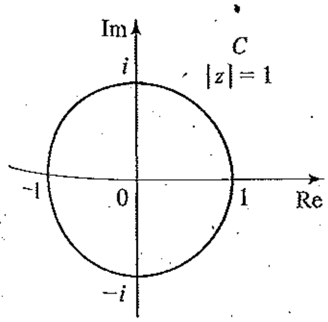


Figure 5

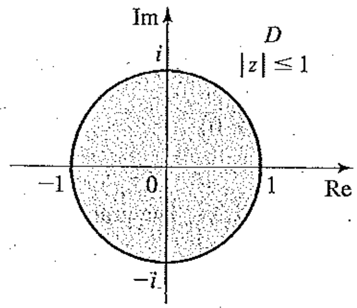


Figure 6

(b) D is the set of complex numbers whose distance from the origin is less than or equal to 1. Thus, D is the disk that consists of all complex numbers on and inside the circle C of part (a), as shown in Figure 6.

Polar Form of Complex Numbers

Let $z = a + bi$ be a complex number, and in the complex plane let's draw the line segment joining the origin to the point $a + bi$ (see Figure 7). The length of this line segment is $r = |z| = \sqrt{a^2 + b^2}$. If θ is an angle in standard position whose terminal side coincides with this line segment, then by the definitions of sine and cosine (see Section 6.2)

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

so $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. We have shown the following.

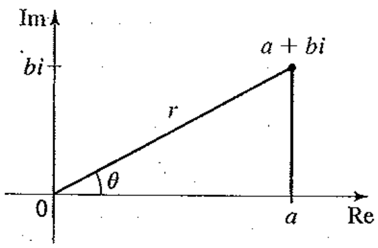


Figure 7

Polar Form of Complex Numbers

A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

The argument of z is not unique, but any two arguments of z differ by a multiple of 2π.

Example 5 Writing Complex Numbers in Polar Form

Write each complex number in trigonometric form.

(a) $1 + i$ (b) $-1 + \sqrt{3}i$ (c) $-4\sqrt{3} - 4i$ (d) $3 + 4i$

Solution These complex numbers are graphed in Figure 8, which helps us find their arguments.

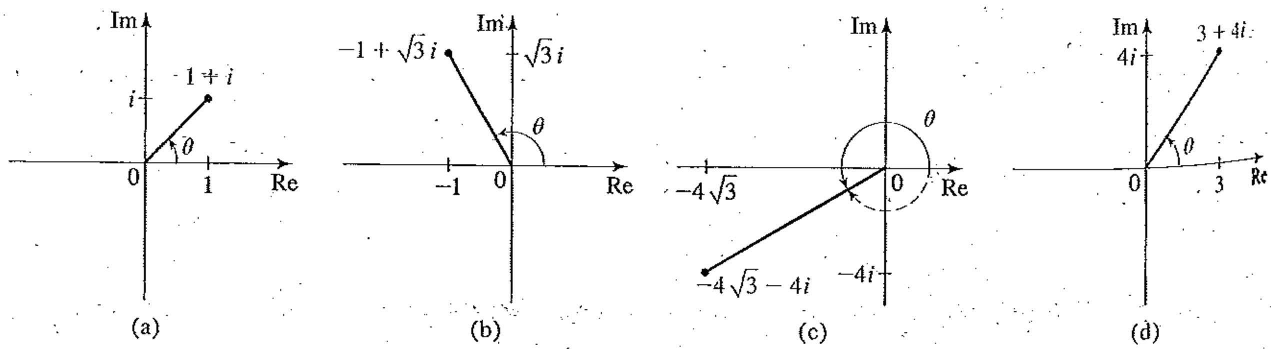


Figure 8

$\tan\theta = \frac{1}{1} = 1$

$\theta = \frac{\pi}{4}$

(a) An argument is $\theta = \pi/4$ and $r = \sqrt{1+1} = \sqrt{2}$. Thus

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

$\theta = \frac{2\pi}{3}$

(b) An argument is $\theta = 2\pi/3$ and $r = \sqrt{1+3} = 2$. Thus

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$\tan\theta = \frac{-4}{-4\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\theta = \frac{7\pi}{6}$

(c) An argument is $\theta = 7\pi/6$ (or we could use $\theta = -5\pi/6$), and $r = \sqrt{48+16} = 8$. Thus

$$-4\sqrt{3} - 4i = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\frac{4}{3}$

(d) An argument is $\theta = \tan^{-1}\frac{4}{3}$ and $r = \sqrt{3^2 + 4^2} = 5$. So

$$3 + 4i = 5\left[\cos\left(\tan^{-1}\tfrac{4}{3}\right) + i\sin\left(\tan^{-1}\tfrac{4}{3}\right)\right]$$ ■

The addition formulas for sine and cosine that we discussed in Section 7.2 greatly simplify the multiplication and division of complex numbers in polar form. The following theorem shows how.

Multiplication and Division of Complex Numbers

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Multiplication}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (z_2 \neq 0) \qquad \text{Division}$$

This theorem says:

To multiply two complex numbers, multiply the moduli and add the arguments.

To divide two complex numbers, divide the moduli and subtract the arguments.

■ **Proof** To prove the multiplication formula, we simply multiply the two complex numbers.

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

In the last step we used the addition formulas for sine and cosine. ■

The proof of the division formula is left as an exercise.

Mathematics in the Modern World

Bill Ross/Corbis

Fractals

Many of the things we model in this book have regular predictable shapes. But recent advances in mathematics have made it possible to model such seemingly random or even chaotic shapes as those of a cloud, a flickering flame, a mountain, or a jagged coastline. The basic tools in this type of modeling are the fractals invented by the mathematician Benoit Mandelbrot. A *fractal* is a geometric shape built up from a simple basic shape by scaling and repeating the shape indefinitely according to a given rule. Fractals have infinite detail; this means the closer you look, the more you see. They are also *self-similar*; that is, zooming in on a portion of the fractal yields the same detail as the original shape. Because of their beautiful shapes, fractals are used by movie makers to create fictional landscapes and exotic backgrounds.

Although a fractal is a complex shape, it is produced according to very simple rules (see page 605). This property of fractals is exploited in a process of storing pictures on a computer called *fractal image compression*. In this process a picture is stored as a simple basic shape and a rule; repeating the shape according to the rule produces the original picture. This is an extremely efficient method of storage; that's how thousands of color pictures can be put on a single compact disc.

Example 6 Multiplying and Dividing Complex Numbers

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find (a) z_1z_2 and (b) z_1/z_2.

Solution

(a) By the multiplication formula

$$\begin{aligned} z_1z_2 &= (2)(5)\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right] \\ &= 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \end{aligned}$$

To approximate the answer, we use a calculator in radian mode and get

$$z_1z_2 \approx 10(-0.2588 + 0.9659i) = -2.588 + 9.659i$$

(b) By the division formula

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{5}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right] \\ &= \frac{2}{5}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right] \\ &= \frac{2}{5}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) \end{aligned}$$

Using a calculator in radian mode, we get the approximate answer:

$$\frac{z_1}{z_2} \approx \tfrac{2}{5}(0.9659 - 0.2588i) = 0.3864 - 0.1035i$$

■

DeMoivre's Theorem

Repeated use of the multiplication formula gives the following useful formula for raising a complex number to a power n for any positive integer n.

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This theorem says: *To take the nth power of a complex number, we take the nth power of the modulus and multiply the argument by n.*

■ **Proof** By the multiplication formula

$$\begin{aligned} z^2 = zz &= r^2[\cos(\theta+\theta) + i\sin(\theta+\theta)] \\ &= r^2(\cos 2\theta + i\sin 2\theta) \end{aligned}$$

Now we multiply z^2 by z to get

$$z^3 = z^2z = r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$$
$$= r^3(\cos 3\theta + i\sin 3\theta)$$

Repeating this argument, we see that for any positive integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

A similar argument using the division formula shows that this also holds for negative integers. ■

Example 7 Finding a Power Using DeMoivre's Theorem

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

Solution Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 5(a) that

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by DeMoivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$
$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$ ■

nth Roots of Complex Numbers

An **nth root** of a complex number z is any complex number w such that $w^n = z$. DeMoivre's Theorem gives us a method for calculating the nth roots of any complex number.

nth Roots of Complex Numbers

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.

■ **Proof** To find the nth roots of z, we need to find a complex number w such that

$$w^n = z$$

Let's write z in polar form:

$$z = r(\cos\theta + i\sin\theta)$$

One nth root of z is

$$w = r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

since by DeMoivre's Theorem, $w^n = z$. But the argument θ of z can be replaced by $\theta + 2k\pi$ for any integer k. Since this expression gives a different value of w for $k = 0$, $1, 2, \ldots, n-1$, we have proved the formula in the theorem. ∎

The following observations help us use the preceding formula.

1. The modulus of each nth root is $r^{1/n}$.
2. The argument of the first root is θ/n.
3. We repeatedly add $2\pi/n$ to get the argument of each successive root.

These observations show that, when graphed, the nth roots of z are spaced equally on the circle of radius $r^{1/n}$.

Example 8 Finding Roots of a Complex Number

Find the six sixth roots of $z = -64$, and graph these roots in the complex plane.

Solution In polar form, $z = 64(\cos \pi + i \sin \pi)$. Applying the formula for nth roots with $n = 6$, we get

$$w_k = 64^{1/6}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i \sin\left(\frac{\pi + 2k\pi}{6}\right)\right]$$

for $k = 0, 1, 2, 3, 4, 5$. Using $64^{1/6} = 2$, we find that the six sixth roots of -64 are

We add $2\pi/6 = \pi/3$ to each argument to get the argument of the next root.

$$w_0 = 64^{1/6}\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$w_1 = 64^{1/6}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 2i$$

$$w_2 = 64^{1/6}\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$$

$$w_3 = 64^{1/6}\left(\cos\frac{7\pi}{6} + i \sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$$

$$w_4 = 64^{1/6}\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right) = -2i$$

$$w_5 = 64^{1/6}\left(\cos\frac{11\pi}{6} + i \sin\frac{11\pi}{6}\right) = \sqrt{3} - i$$

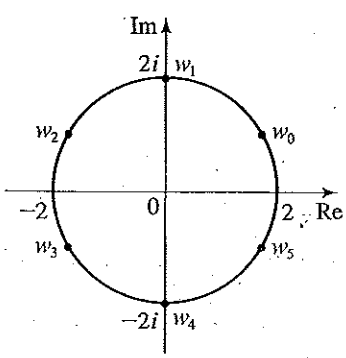


Figure 9
The six sixth roots of $z = -64$

All these points lie on a circle of radius 2, as shown in Figure 9. ∎

When finding roots of complex numbers, we sometimes write the argument θ of the complex number in degrees. In this case, the nth roots are obtained from the formula

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i \sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

Example 9 Finding Cube Roots of a Complex Number

Find the three cube roots of $z = 2 + 2i$, and graph these roots in the complex plane.

Solution First we write z in polar form using degrees. We have $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $\theta = 45°$. Thus

$$z = 2\sqrt{2}(\cos 45° + i \sin 45°)$$

Applying the formula for nth roots (in degrees) with $n = 3$, we find the cube roots of z are of the form

$(2\sqrt{2})^{1/3} = (2^{3/2})^{1/3} = 2^{1/2} = \sqrt{2}$

We add $360°/3 = 120°$ to each argument to get the argument of the next root.

$$w_k = (2\sqrt{2})^{1/3}\left[\cos\left(\frac{45° + 360°k}{3}\right) + i \sin\left(\frac{45° + 360°k}{3}\right)\right]$$

where $k = 0, 1, 2$. Thus, the three cube roots are

$$w_0 = \sqrt{2}(\cos 15° + i \sin 15°) \approx 1.366 + 0.366i$$
$$w_1 = \sqrt{2}(\cos 135° + i \sin 135°) = -1 + i$$
$$w_2 = \sqrt{2}(\cos 255° + i \sin 255°) \approx -0.366 - 1.366i$$

The three cube roots of z are graphed in Figure 10. These roots are spaced equally on a circle of radius $\sqrt{2}$. ■

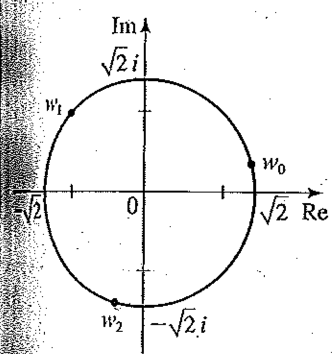


Figure 10
The three cube roots of $z = 2 + 2i$

Example 10 Solving an Equation Using the nth Roots Formula

Solve the equation $z^6 + 64 = 0$.

Solution This equation can be written as $z^6 = -64$. Thus, the solutions are the sixth roots of -64, which we found in Example 8. ■

8.3 Exercises

1–8 ■ Graph the complex number and find its modulus.

1. $4i$
2. $-3i$
3. -2
4. 6
5. $5 + 2i$
6. $7 - 3i$
7. $\sqrt{3} + i$
8. $-1 - \frac{\sqrt{3}}{3}i$
9. $\frac{3 + 4i}{5}$
10. $\frac{-\sqrt{2} + i\sqrt{2}}{2}$

11–12 ■ Sketch the complex number z, and also sketch $2z$, $-z$, and $\frac{1}{2}z$ on the same complex plane.

11. $z = 1 + i$
12. $z = -1 + i\sqrt{3}$

13–14 ■ Sketch the complex number z and its complex conjugate $\bar{z}$ on the same complex plane.

13. $z = 8 + 2i$
14. $z = -5 + 6i$

15–16 ■ Sketch z_1, z_2, $z_1 + z_2$, and z_1z_2 on the same complex plane.

15. $z_1 = 2 - i, \quad z_2 = 2 + i$
16. $z_1 = -1 + i, \quad z_2 = 2 - 3i$

17–24 ■ Sketch the set in the complex plane.

17. $\{z = a + bi \mid a \le 0, b \ge 0\}$
18. $\{z = a + bi \mid a > 1, b > 1\}$
19. $\{z \mid |z| = 3\}$
20. $\{z \mid |z| \ge 1\}$
21. $\{z \mid |z| < 2\}$
22. $\{z \mid 2 \le |z| \le 5\}$
23. $\{z = a + bi \mid a + b < 2\}$
24. $\{z = a + bi \mid a \ge b\}$

25–48 ■ Write the complex number in polar form with argument θ between 0 and 2π.

25. $1 + i$ **26.** $1 + \sqrt{3}\, i$ **27.** $\sqrt{2} - \sqrt{2}\, i$

28. $1 - i$ **29.** $2\sqrt{3} - 2i$ **30.** $-1 + i$

31. $-3i$ **32.** $-3 - 3\sqrt{3}\, i$ **33.** $5 + 5i$

34. 4 **35.** $4\sqrt{3} - 4i$ **36.** $8i$

37. -20 **38.** $\sqrt{3} + i$ **39.** $3 + 4i$

40. $i(2 - 2i)$ **41.** $3i(1 + i)$ **42.** $2(1 - i)$

43. $4(\sqrt{3} + i)$ **44.** $-3 - 3i$ **45.** $2 + i$

46. $3 + \sqrt{3}\, i$ **47.** $\sqrt{2} + \sqrt{2}\, i$ **48.** $-\pi i$

49–56 ■ Find the product $z_1 z_2$ and the quotient z_1/z_2. Express your answer in polar form.

49. $z_1 = \cos \pi + i \sin \pi, \quad z_2 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$

50. $z_1 = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}, \quad z_2 = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$

51. $z_1 = 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right), \quad z_2 = 5\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

52. $z_1 = 7\left(\cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8}\right), \quad z_2 = 2\left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}\right)$

53. $z_1 = 4(\cos 120^\circ + i \sin 120^\circ)$,
$z_2 = 2(\cos 30^\circ + i \sin 30^\circ)$

54. $z_1 = \sqrt{2}(\cos 75^\circ + i \sin 75^\circ)$,
$z_2 = 3\sqrt{2}(\cos 60^\circ + i \sin 60^\circ)$

55. $z_1 = 4(\cos 200^\circ + i \sin 200^\circ)$,
$z_2 = 25(\cos 150^\circ + i \sin 150^\circ)$

56. $z_1 = \frac{4}{5}(\cos 25^\circ + i \sin 25^\circ)$,
$z_2 = \frac{1}{5}(\cos 155^\circ + i \sin 155^\circ)$

57–64 ■ Write z_1 and z_2 in polar form, and then find the product $z_1 z_2$ and the quotients z_1/z_2 and $1/z_1$.

57. $z_1 = \sqrt{3} + i, \quad z_2 = 1 + \sqrt{3}\, i$

58. $z_1 = \sqrt{2} - \sqrt{2}\, i, \quad z_2 = 1 - i$

59. $z_1 = 2\sqrt{3} - 2i, \quad z_2 = -1 + i$

60. $z_1 = -\sqrt{2}\, i, \quad z_2 = -3 - 3\sqrt{3}\, i$

61. $z_1 = 5 + 5i, \quad z_2 = 4$ **62.** $z_1 = 4\sqrt{3} - 4i, \quad z_2 = 8i$

63. $z_1 = -20, \quad z_2 = \sqrt{3} + i$ **64.** $z_1 = 3 + 4i, \quad z_2 = 2 - 2i$

65–76 ■ Find the indicated power using DeMoivre's Theorem.

65. $(1 + i)^{20}$ **66.** $(1 - \sqrt{3}\, i)^5$

67. $(2\sqrt{3} + 2i)^5$ **68.** $(1 - i)^8$

69. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^{12}$ **70.** $(\sqrt{3} - i)^{-10}$

71. $(2 - 2i)^8$ **72.** $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{15}$

73. $(-1 - i)^7$ **74.** $(3 + \sqrt{3}\, i)^4$

75. $(2\sqrt{3} + 2i)^{-5}$ **76.** $(1 - i)^{-8}$

77–86 ■ Find the indicated roots, and graph the roots in the complex plane.

77. The square roots of $4\sqrt{3} + 4i$

78. The cube roots of $4\sqrt{3} + 4i$

79. The fourth roots of $-81i$ **80.** The fifth roots of 32

81. The eighth roots of 1 **82.** The cube roots of $1 + i$

83. The cube roots of i **84.** The fifth roots of i

85. The fourth roots of -1

86. The fifth roots of $-16 - 16\sqrt{3}i$

87–92 ■ Solve the equation.

87. $z^4 + 1 = 0$ **88.** $z^8 - i = 0$

89. $z^3 - 4\sqrt{3} - 4i = 0$ **90.** $z^6 - 1 = 0$

91. $z^3 + 1 = -i$ **92.** $z^3 - 1 = 0$

93. **(a)** Let $w = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ where n is a positive integer. Show that $1, w, w^2, w^3, \ldots, w^{n-1}$ are the n distinct nth roots of 1.

(b) If $z \neq 0$ is any complex number and $s^n = z$, show that the n distinct nth roots of z are

$$s, sw, sw^2, sw^3, \ldots, sw^{n-1}$$

Discovery • Discussion

94. Sums of Roots of Unity Find the exact values of all three cube roots of 1 (see Exercise 93) and then add them. Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the sum of the nth roots of 1, for any n?

95. Products of Roots of Unity Find the product of the three cube roots of 1 (see Exercise 93). Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the product of the nth roots of 1, for any n?

96. Complex Coefficients and the Quadratic Formula The quadratic formula works whether the coefficients of the equation are real or complex. Solve these equations using the quadratic formula, and, if necessary, DeMoivre's Theorem.

(a) $z^2 + (1 + i)z + i = 0$

(b) $z^2 - iz + 1 = 0$

(c) $z^2 - (2 - i)z - \frac{1}{4}i = 0$