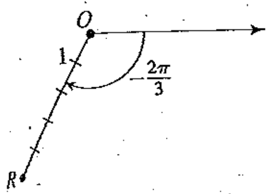
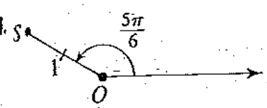
23–24 ■ A point is graphed in polar form. Find its rectangular coordinates.

23.



24.



25–32 ■ Find the rectangular coordinates for the point whose polar coordinates are given.

25. $(4, \pi/6)$
26. $(6, 2\pi/3)$
27. $(\sqrt{2}, -\pi/4)$
28. $(-1, 5\pi/2)$
29. $(5, 5\pi)$
30. $(0, 13\pi)$
31. $(6\sqrt{2}, 11\pi/6)$
32. $(\sqrt{3}, -5\pi/3)$

33–40 ■ Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$.

33. $(-1, 1)$
34. $(3\sqrt{3}, -3)$
35. $(\sqrt{8}, \sqrt{8})$
36. $(-\sqrt{6}, -\sqrt{2})$
37. $(3, 4)$
38. $(1, -2)$
39. $(-6, 0)$
40. $(0, -\sqrt{3})$

41–46 ■ Convert the equation to polar form.

41. $x = y$
42. $x^2 + y^2 = 9$
43. $y = x^2$
44. $y = 5$
45. $x = 4$
46. $x^2 - y^2 = 1$

47–60 ■ Convert the polar equation to rectangular coordinates.

47. $r = 7$
48. $\theta = \pi$
49. $r \cos\theta = 6$
50. $r = 6\cos\theta$
51. $r^2 = \tan\theta$
52. $r^2 = \sin 2\theta$
53. $r = \dfrac{1}{\sin\theta - \cos\theta}$
54. $r = \dfrac{1}{1 + \sin\theta}$
55. $r = 1 + \cos\theta$
56. $r = \dfrac{4}{1 + 2\sin\theta}$
57. $r = 2\sec\theta$
58. $r = 2 - \cos\theta$
59. $\sec\theta = 2$
60. $\cos 2\theta = 1$

Discovery • Discussion

61. The Distance Formula in Polar Coordinates

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(1, 7\pi/6)$, using the formula from part (a).

(c) Now convert the points in part (b) to rectangular coordinates. Find the distance between them using the usual Distance Formula. Do you get the same answer?

8.2 Graphs of Polar Equations

The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations than rectangular equations.

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a) on the next page). To plot points in polar coordinates, it is convenient to use a grid consisting of circles centered at the pole and rays emanating from the pole, as in Figure 1(b). We will use such grids to help us sketch polar graphs.

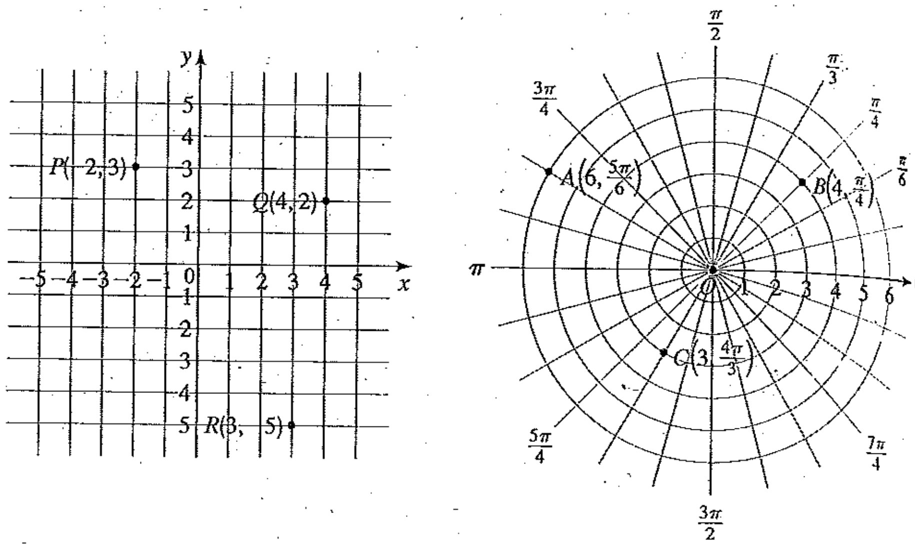


Figure 1 (a) Grid for rectangular coordinates (b) Grid for polar coordinates

In Examples 1 and 2 we see that circles centered at the origin and lines that pass through the origin have particularly simple equations in polar coordinates.

Example 1 Sketching the Graph of a Polar Equation

Sketch the graph of the equation $r = 3$ and express the equation in rectangular coordinates.

Solution The graph consists of all points whose r-coordinate is 3, that is, all points that are 3 units away from the origin. So the graph is a circle of radius 3 centered at the origin, as shown in Figure 2.

Squaring both sides of the equation, we get

$$r^2 = 3^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 = 9 \qquad \text{Substitute } r^2 = x^2 + y^2$$

So the equivalent equation in rectangular coordinates is $x^2 + y^2 = 9$. ∎

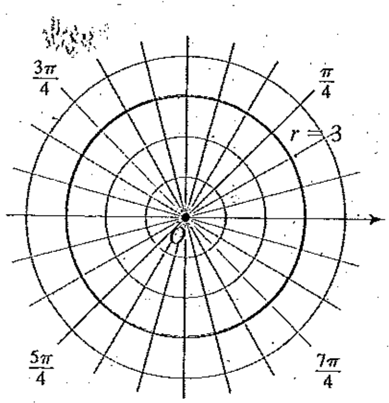


Figure 2

In general, the graph of the equation $r = a$ is a circle of radius $|a|$ centered at the origin. Squaring both sides of this equation, we see that the equivalent equation in rectangular coordinates is $x^2 + y^2 = a^2$.

Example 2 Sketching the Graph of a Polar Equation

Sketch the graph of the equation $\theta = \pi/3$ and express the equation in rectangular coordinates.

Solution The graph consists of all points whose θ-coordinate is $\pi/3$. This is the straight line that passes through the origin and makes an angle of $\pi/3$ with the polar

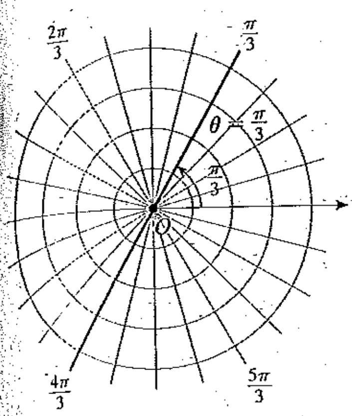


Figure 3

axis (see Figure 3). Note that the points $(r, \pi/3)$ on the line with $r > 0$ lie in quadrant I, whereas those with $r < 0$ lie in quadrant III. If the point (x, y) lies on this line, then

$$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$$

Thus, the rectangular equation of this line is $y = \sqrt{3}\,x$. ■

To sketch a polar curve whose graph isn't as obvious as the ones in the preceding examples, we plot points calculated for sufficiently many values of θ and then join them in a continuous curve. (This is what we did when we first learned to graph functions in rectangular coordinates.)

Example 3 Sketching the Graph of a Polar Equation

Sketch the graph of the polar equation $r = 2\sin\theta$.

Solution We first use the equation to determine the polar coordinates of several points on the curve. The results are shown in the following table.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$r = 2\sin\theta$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

We plot these points in Figure 4 and then join them to sketch the curve. The graph appears to be a circle. We have used values of θ only between 0 and π, since the same points (this time expressed with negative r-coordinates) would be obtained if we allowed θ to range from π to 2π.

The polar equation $r = 2\sin\theta$ in rectangular coordinates is

$$x^2 + (y - 1)^2 = 1$$

(See Section 8.1, Example 6(b)). From the rectangular form of the equation we see that the graph is a circle of radius 1 centered at $(0, 1)$.

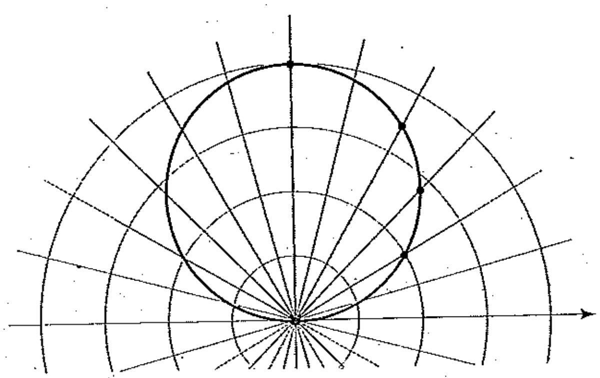


Figure 4
$r = 2\sin\theta$

■

In general, the graphs of equations of the form

$$r = 2a\sin\theta \qquad \text{and} \qquad r = 2a\cos\theta$$

are circles with radius $|a|$ centered at the points with polar coordinates $(a, \pi/2)$ and $(a, 0)$, respectively.

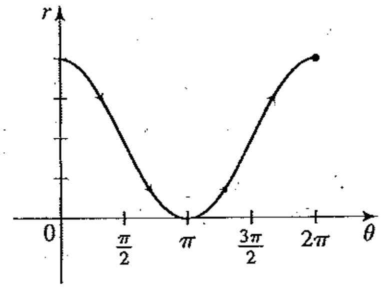


Figure 5
$r = 2 + 2\cos\theta$

Example 4 Sketching the Graph of a Polar Equation

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution Instead of plotting points as in Example 3, we first sketch the graph of $r = 2 + 2\cos\theta$ in *rectangular* coordinates in Figure 5. We can think of this graph as a table of values that enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) decreases from 4 to 2, so we sketch the corresponding part of the polar graph in Figure 6(a). As θ increases from $\pi/2$ to π, Figure 5 shows that r decreases from 2 to 0, so we sketch the next part of the graph as in Figure 6(b). As θ increases from π to $3\pi/2$, r increases from 0 to 2, as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 2 to 4, as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Combining the portions of the graph from parts (a) through (d) of Figure 6, we sketch the complete graph in part (e).

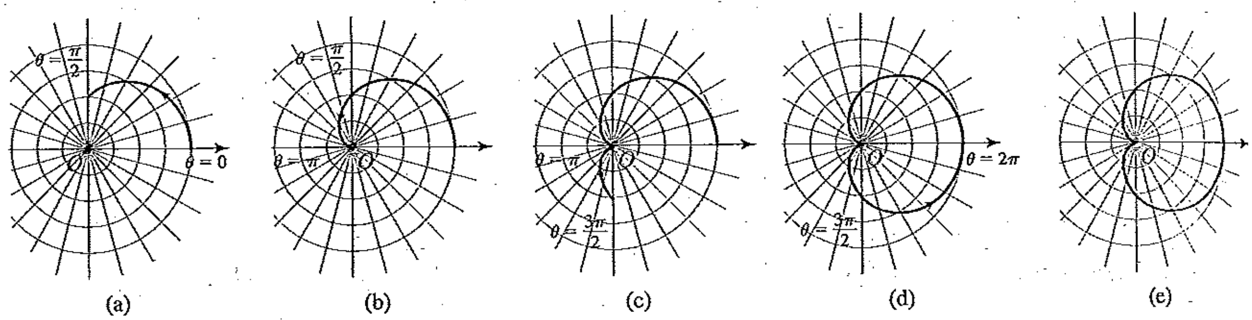


Figure 6 Steps in sketching $r = 2 + 2\cos\theta$ ∎

The polar equation $r = 2 + 2\cos\theta$ in rectangular coordinates is

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

(See Section 8.1, Example 6(c)). The simpler form of the polar equation shows that it is more natural to describe cardioids using polar coordinates.

The curve in Figure 6 is called a **cardioid** because it is heart-shaped. In general, the graph of any equation of the form

$$r = a(1 \pm \cos\theta) \qquad \text{or} \qquad r = a(1 \pm \sin\theta)$$

is a cardioid.

Example 5 Sketching the Graph of a Polar Equation

Sketch the curve $r = \cos 2\theta$.

Solution As in Example 4, we first sketch the graph of $r = \cos 2\theta$ in *rectangular* coordinates, as shown in Figure 7. As θ increases from 0 to $\pi/4$, Figure 7 shows that r decreases from 1 to 0, and so we draw the corresponding portion of the polar curve in Figure 8 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, the value of r goes from 0 to -1. This means that the distance from the origin increases from 0 to 1, but instead of being in quadrant I, this portion of the polar curve (indicated by ②) lies on the opposite side of the origin in quadrant III. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four petals and is called a **four-leaved rose.**

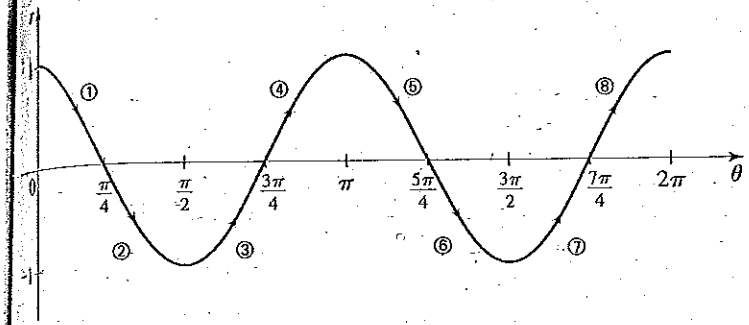


Figure 7
Graph of $r = \cos 2\theta$ sketched in rectangular coordinates

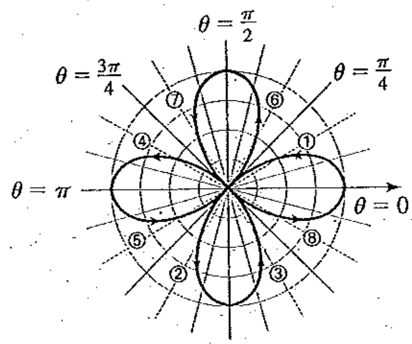


Figure 8
Four-leaved rose $r = \cos 2\theta$ sketched in polar coordinates ■

In general, the graph of an equation of the form

$$r = a \cos n\theta \qquad \text{or} \qquad r = a \sin n\theta$$

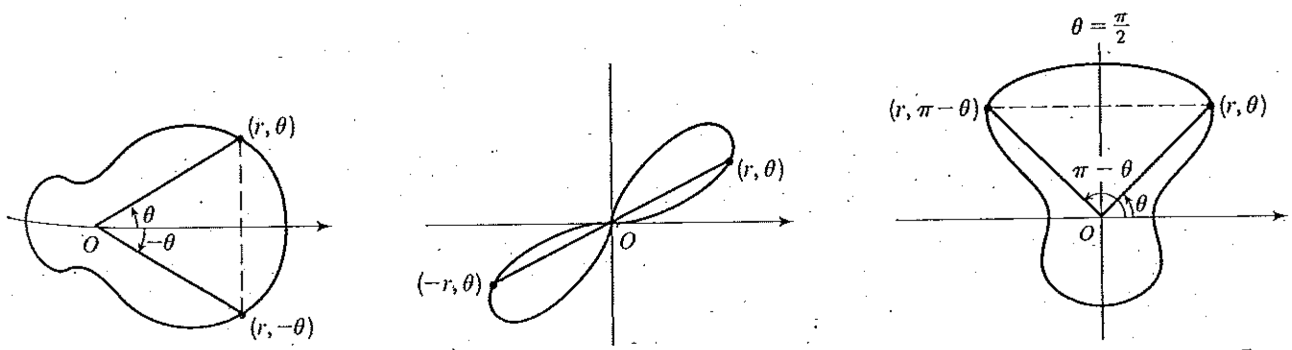
is an **n-leaved rose** if n is odd or a $2n$-leaved rose if n is even (as in Example 5).

Symmetry

When graphing a polar equation, it's often helpful to take advantage of symmetry. We list three tests for symmetry; Figure 9 shows why these tests work.

Tests for Symmetry

1. If a polar equation is unchanged when we replace θ by $-\theta$, then the graph is symmetric about the polar axis (Figure 9(a)).
2. If the equation is unchanged when we replace r by $-r$, then the graph is symmetric about the pole (Figure 9(b)).
3. If the equation is unchanged when we replace θ by $\pi - \theta$, the graph is symmetric about the vertical line $\theta = \pi/2$ (the y-axis) (Figure 9(c)).



(a) Symmetry about the polar axis (b) Symmetry about the pole (c) Symmetry about the line $\theta = \frac{\pi}{2}$

Figure 9

The graphs in Figures 2, 6(e), and 8 are symmetric about the polar axis. The graph in Figure 8 is also symmetric about the pole. Figures 4 and 8 show graphs that are symmetric about $\theta = \pi/2$. Note that the four-leaved rose in Figure 8 meets all three tests for symmetry.

In rectangular coordinates, the zeros of the function $y = f(x)$ correspond to the x-intercepts of the graph. In polar coordinates, the zeros of the function $r = f(\theta)$ are the angles θ at which the curve crosses the pole. The zeros help us sketch the graph, as illustrated in the next example.

Example 6 Using Symmetry to Sketch a Polar Graph

Sketch the graph of the equation $r = 1 + 2\cos\theta$.

Solution We use the following as aids in sketching the graph.

- **Symmetry** Since the equation is unchanged when θ is replaced by $-\theta$, the graph is symmetric about the polar axis.
- **Zeros** To find the zeros, we solve

$$0 = 1 + 2\cos\theta$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

- **Table of values** As in Example 4, we sketch the graph of $r = 1 + 2\cos\theta$ in *rectangular* coordinates to serve as a table of values (Figure 10).

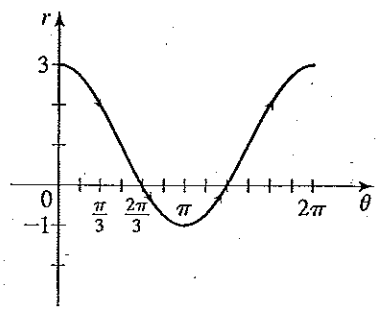


Figure 10

Now we sketch the polar graph of $r = 1 + 2\cos\theta$ from $\theta = 0$ to $\theta = \pi$, and then use symmetry to complete the graph in Figure 11. ■

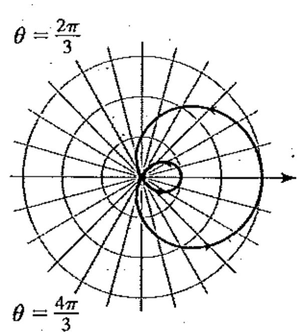


Figure 11
$r = 1 + 2\cos\theta$

The curve in Figure 11 is called a **limaçon**, after the Middle French word for snail. In general, the graph of an equation of the form

$$r = a \pm b\cos\theta \quad \text{or} \quad r = a \pm b\sin\theta$$

is a limaçon. The shape of the limaçon depends on the relative size of a and b (see the table on page 594).

Graphing Polar Equations with Graphing Devices

Although it's useful to be able to sketch simple polar graphs by hand, we need a graphing calculator or computer when the graph is as complicated as the one in Figure 12. Fortunately, most graphing calculators are capable of graphing polar equations directly.

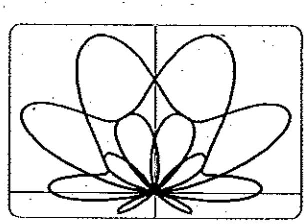


Figure 12
$r = \sin\theta + \sin^3(5\theta/2)$

Example 7 Drawing the Graph of a Polar Equation

Graph the equation $r = \cos(2\theta/3)$.

Solution We need to determine the domain for θ. So we ask ourselves: How many complete rotations are required before the graph starts to repeat itself? The graph repeats itself when the same value of r is obtained at θ and $\theta + 2n\pi$. Thus, we need to find an integer n, so that

$$\cos\frac{2(\theta + 2n\pi)}{3} = \cos\frac{2\theta}{3}$$

For this equality to hold, $4n\pi/3$ must be a multiple of 2π, and this first happens when $n = 3$. Therefore, we obtain the entire graph if we choose values of θ between $\theta = 0$ and $\theta = 0 + 2(3)\pi = 6\pi$. The graph is shown in Figure 13.

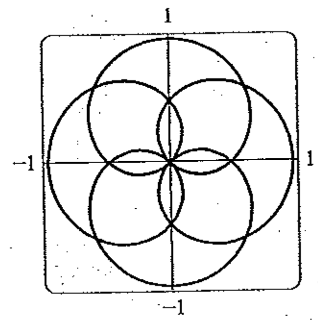


Figure 13
$r = \cos(2\theta/3)$

Example 8 A Family of Polar Equations

Graph the family of polar equations $r = 1 + c\sin\theta$ for $c = 3, 2.5, 2, 1.5, 1$. How does the shape of the graph change as c changes?

Solution Figure 14 shows computer-drawn graphs for the given values of c. For $c > 1$, the graph has an inner loop; the loop decreases in size as c decreases. When $c = 1$, the loop disappears and the graph becomes a cardioid (see Example 4).

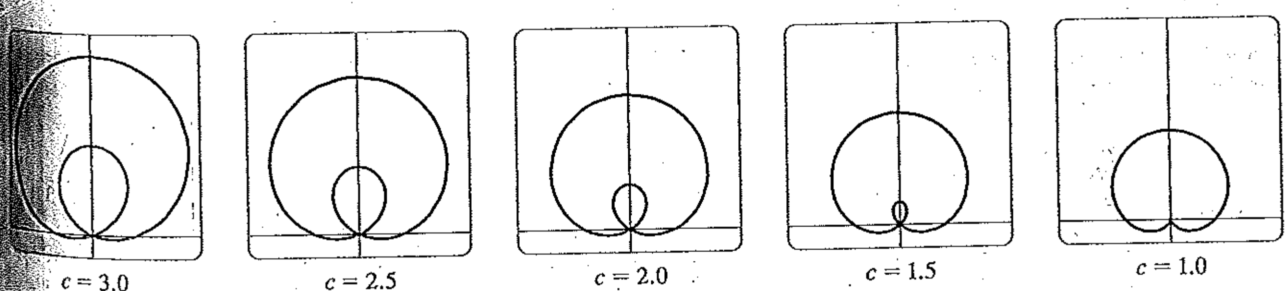
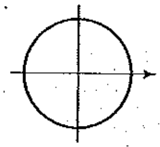


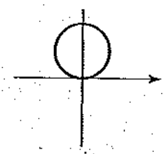
Figure 14 A family of limaçons $r = 1 + c\sin\theta$ in the viewing rectangle $[-2.5, 2.5]$ by $[-0.5, 4.5]$

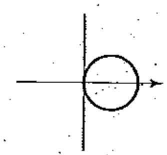
The following box gives a summary of some of the basic polar graphs used in calculus.

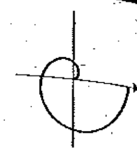
Some Common Polar Curves

Circles and Spiral


$r = a$
circle


$r = a \sin \theta$
circle


$r = a \cos \theta$
circle


$r = a\theta$
spiral

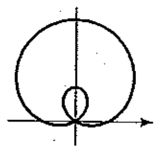
Limaçons

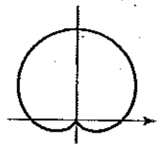
$r = a \pm b \sin \theta$

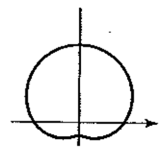
$r = a \pm b \cos \theta$

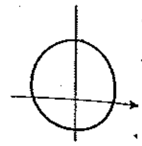
$(a > 0, b > 0)$

Orientation depends on the trigonometric function (sine or cosine) and the sign of b.


$a < b$
limaçon with inner loop


$a = b$
cardioid


$a > b$
dimpled limaçon


$a \geq 2b$
convex limaçon

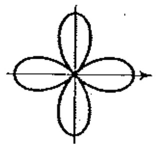
Roses

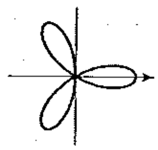
$r = a \sin n\theta$

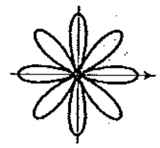
$r = a \cos n\theta$

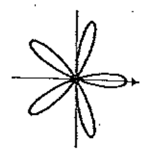
n-leaved if n is odd

$2n$-leaved if n is even


$r = a \cos 2\theta$
4-leaved rose

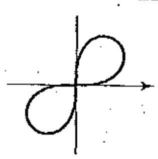

$r = a \cos 3\theta$
3-leaved rose

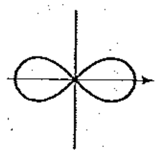

$r = a \cos 4\theta$
8-leaved rose


$r = a \cos 5\theta$
5-leaved rose

Lemniscates

Figure-eight-shaped curves


$r^2 = a^2 \sin 2\theta$
lemniscate

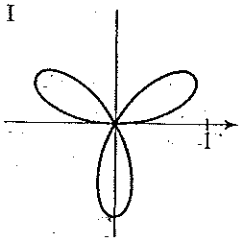

$r^2 = a^2 \cos 2\theta$
lemniscate

8.2 Exercises

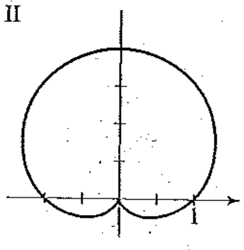
1–6 ■ Match the polar equation with the graphs labeled I–VI. Use the table above to help you.

1. $r = 3 \cos \theta$
2. $r = 3$
3. $r = 2 + 2 \sin \theta$
4. $r = 1 + 2 \cos \theta$
5. $r = \sin 3\theta$
6. $r = \sin 4\theta$

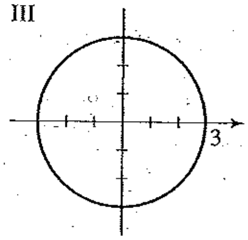
I



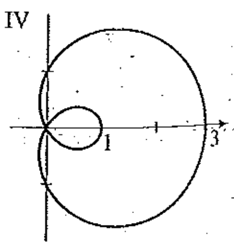
II



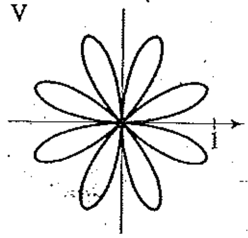
III



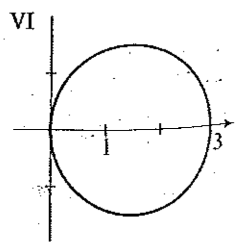
IV



V



VI



7–14 ■ Test the polar equation for symmetry with respect to the polar axis, the pole, and the line $\theta = \pi/2$.

7. $r = 2 - \sin\theta$
8. $r = 4 + 8\cos\theta$
9. $r = 3\sec\theta$
10. $r = 5\cos\theta\csc\theta$
11. $r = \dfrac{4}{3 - 2\sin\theta}$
12. $r = \dfrac{5}{1 + 3\cos\theta}$
13. $r^2 = 4\cos 2\theta$
14. $r^2 = 9\sin\theta$

15–36 ■ Sketch the graph of the polar equation.

15. $r = 2$
16. $r = -1$
17. $\theta = -\pi/2$
18. $\theta = 5\pi/6$
19. $r = 6\sin\theta$
20. $r = \cos\theta$
21. $r = -2\cos\theta$
22. $r = 2\sin\theta + 2\cos\theta$
23. $r = 2 - 2\cos\theta$
24. $r = 1 + \sin\theta$
25. $r = -3(1 + \sin\theta)$
26. $r = \cos\theta - 1$
27. $r = \theta,\ \ \theta \geq 0$ (spiral)
28. $r\theta = 1,\ \ \theta > 0$ (reciprocal spiral)
29. $r = \sin 2\theta$ (four-leaved rose)
30. $r = 2\cos 3\theta$ (three-leaved rose)
31. $r^2 = \cos 2\theta$ (lemniscate)
32. $r^2 = 4\sin 2\theta$ (lemniscate)
33. $r = 2 + \sin\theta$ (limaçon)
34. $r = 1 - 2\cos\theta$ (limaçon)
35. $r = 2 + \sec\theta$ (conchoid)
36. $r = \sin\theta\tan\theta$ (cissoid)

37–40 ■ Use a graphing device to graph the polar equation. Choose the domain of θ to make sure you produce the entire graph.

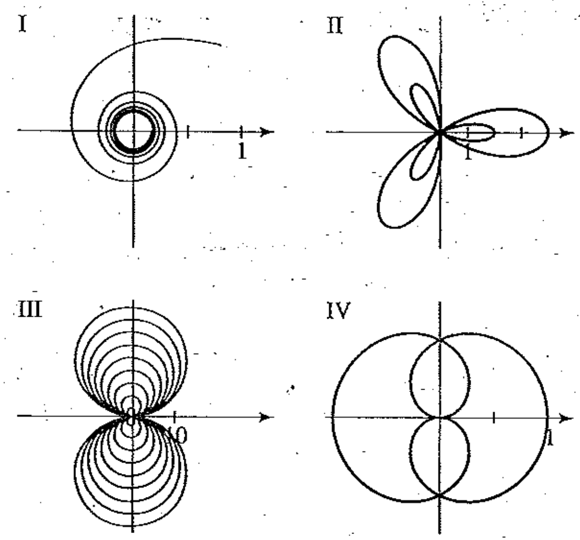
37. $r = \cos(\theta/2)$
38. $r = \sin(8\theta/5)$
39. $r = 1 + 2\sin(\theta/2)$ (nephroid)
40. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

41. Graph the family of polar equations $r = 1 + \sin n\theta$ for $n = 1, 2, 3, 4$, and 5. How is the number of loops related to n?

42. Graph the family of polar equations $r = 1 + c\sin 2\theta$ for $c = 0.3, 0.6, 1, 1.5$, and 2. How does the graph change as c increases?

43–46 ■ Match the polar equation with the graphs labeled I–IV. Give reasons for your answers.

43. $r = \sin(\theta/2)$
44. $r = 1/\sqrt{\theta}$
45. $r = \theta\sin\theta$
46. $r = 1 + 3\cos(3\theta)$



47–50 ■ Sketch a graph of the rectangular equation. [*Hint:* First convert the equation to polar coordinates.]

47. $(x^2 + y^2)^3 = 4x^2y^2$
48. $(x^2 + y^2)^3 = (x^2 - y^2)^2$
49. $(x^2 + y^2)^2 = x^2 - y^2$
50. $x^2 + y^2 = (x^2 + y^2 - x)^2$

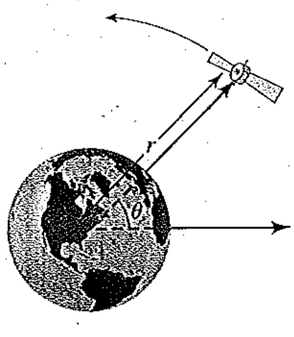
51. Show that the graph of $r = a\cos\theta + b\sin\theta$ is a circle, and find its center and radius.

52. (a) Graph the polar equation $r = \tan\theta\sec\theta$ in the viewing rectangle $[-3, 3]$ by $[-1, 9]$.
 (b) Note that your graph in part (a) looks like a parabola (see Section 2.5). Confirm this by converting the equation to rectangular coordinates.

Applications

53. **Orbit of a Satellite** Scientists and engineers often use polar equations to model the motion of satellites in earth orbit. Let's consider a satellite whose orbit is modeled by the equation $r = 22500/(4 - \cos\theta)$, where r is the distance in miles between the satellite and the center of the earth and θ is the angle shown in the figure on the next page.
 (a) On the same viewing screen, graph the circle $r = 3960$ (to represent the earth, which we will assume to be a sphere of radius 3960 mi) and the polar equation of the satellite's orbit. Describe the motion of the satellite as θ increases from 0 to 2π.

(b) For what angle θ is the satellite closest to the earth? Find the height of the satellite above the earth's surface for this value of θ.



54. **An Unstable Orbit** The orbit described in Exercise 53 is stable because the satellite traverses the same path over and over as θ increases. Suppose that a meteor strikes the satellite and changes its orbit to

$$r = \frac{22500\left(1 - \dfrac{\theta}{40}\right)}{4 - \cos\theta}$$

(a) On the same viewing screen, graph the circle $r = 3960$ and the new orbit equation, with θ increasing from 0 to 3π. Describe the new motion of the satellite.

(b) Use the TRACE feature on your graphing calculator to find the value of θ at the moment the satellite crashes into the earth.

Discovery • Discussion

55. **A Transformation of Polar Graphs** How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

56. **Choosing a Convenient Coordinate System** Compare the polar equation of the circle $r = 2$ with its equation in rectangular coordinates. In which coordinate system is the equation simpler? Do the same for the equation of the four-leaved rose $r = \sin 2\theta$. Which coordinate system would you choose to study these curves?

57. **Choosing a Convenient Coordinate System** Compare the rectangular equation of the line $y = 2$ with its polar equation. In which coordinate system is the equation simpler? Which coordinate system would you choose to study lines?

8.3 Polar Form of Complex Numbers; DeMoivre's Theorem

In this section we represent complex numbers in polar (or trigonometric) form. This enables us to find the nth roots of complex numbers. To describe the polar form of complex numbers, we must first learn to work with complex numbers graphically.

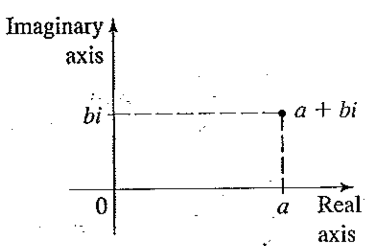


Figure 1

Graphing Complex Numbers

To graph real numbers or sets of real numbers, we have been using the number line, which has just one dimension. Complex numbers, however, have two components: a real part and an imaginary part. This suggests that we need two axes to graph complex numbers: one for the real part and one for the imaginary part. We call these the **real axis** and the **imaginary axis**, respectively. The plane determined by these two axes is called the **complex plane**. To graph the complex number $a + bi$, we plot the ordered pair of numbers (a, b) in this plane, as indicated in Figure 1.

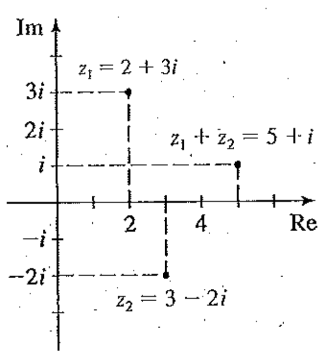


Figure 2

Example 1 Graphing Complex Numbers

Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

Solution We have $z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$. The graph is shown in Figure 2. ∎