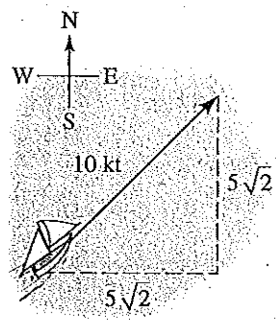


that a boat is sailing at 10 knots to the northeast. We can also express this graphically by drawing an arrow of length 10 in the direction of travel. The velocity can be completely described by the displacement of the arrow from tail to head, which we express as the vector $\langle 5\sqrt{2}, 5\sqrt{2}\rangle$ (see the figure).

In the *Focus on Modeling* (page 630) we will see how polar coordinates are used to draw a (flat) map of a (spherical) world. In the *Discovery Project* on page 626 we explore how an analysis of the vector forces of wind and current can be used to navigate a sailboat.

8.1 Polar Coordinates

In this section we define polar coordinates, and we learn how polar coordinates are related to rectangular coordinates.

Definition of Polar Coordinates

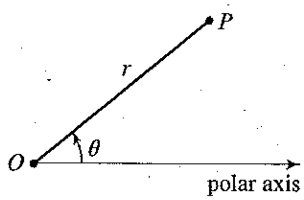


Figure 1

The **polar coordinate system** uses distances and directions to specify the location of a point in the plane. To set up this system, we choose a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray (half-line) called the **polar axis** as in Figure 1. Then each point P can be assigned polar coordinates $P(r, \theta)$ where

r is the *distance* from O to P

θ is the *angle* between the polar axis and the segment $\overline{OP}$

We use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then $P(r, \theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ (see Figure 2).

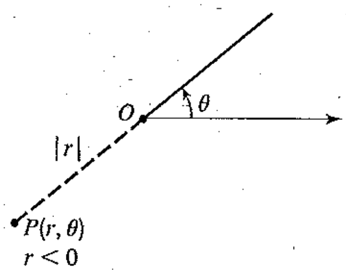


Figure 2

Example 1 Plotting Points in Polar Coordinates

Plot the points whose polar coordinates are given.

(a) $(1, 3\pi/4)$ (b) $(3, -\pi/6)$ (c) $(3, 3\pi)$ (d) $(-4, \pi/4)$

Solution The points are plotted in Figure 3. Note that the point in part (d) lies 4 units from the origin along the angle $5\pi/4$, because the given value of r is negative.

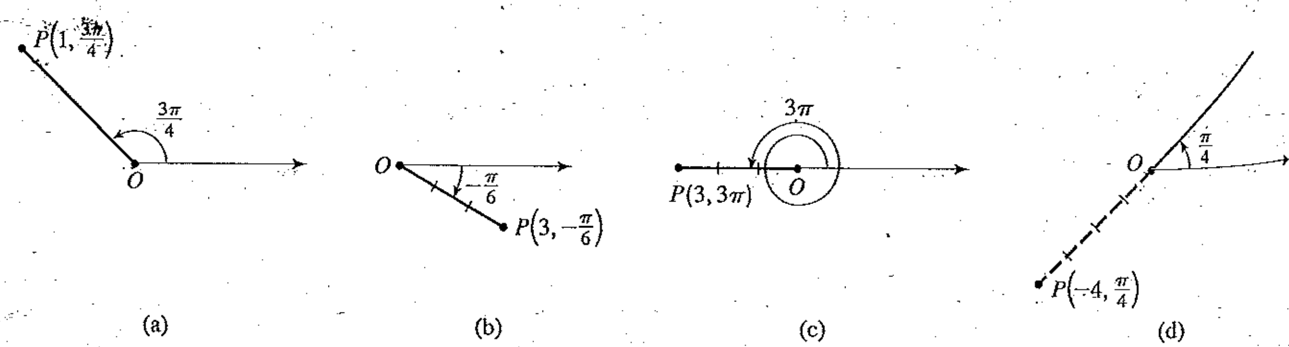


Figure 3

Note that the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point, as shown in Figure 4. Moreover, because the angles $\theta + 2n\pi$ (where n is any integer) all have the same terminal side as the angle θ, each point in the plane has infinitely many representations in polar coordinates. In fact, any point $P(r, \theta)$ can also be represented by

$$P(r, \theta + 2n\pi) \qquad \text{and} \qquad P(-r, \theta + (2n + 1)\pi)$$

for any integer n.

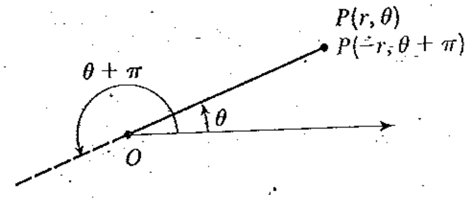


Figure 4

Example 2 Different Polar Coordinates for the Same Point

(a) Graph the point with polar coordinates $P(2, \pi/3)$.

(b) Find two other polar coordinate representations of P with $r > 0$, and two with $r < 0$.

Solution

(a) The graph is shown in Figure 5(a).

(b) Other representations with $r > 0$ are

$$\left(2, \frac{\pi}{3} + 2\pi\right) = \left(2, \frac{7\pi}{3}\right) \qquad \text{Add } 2\pi \text{ to } \theta$$

$$\left(2, \frac{\pi}{3} - 2\pi\right) = \left(2, -\frac{5\pi}{3}\right) \qquad \text{Add } -2\pi \text{ to } \theta$$

Other representations with $r < 0$ are

$$\left(-2, \frac{\pi}{3} + \pi\right) = \left(-2, \frac{4\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } \pi \text{ to } \theta$$

$$\left(-2, \frac{\pi}{3} - \pi\right) = \left(-2, -\frac{2\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } -\pi \text{ to } \theta$$

The graphs in Figure 5 explain why these coordinates represent the same point.

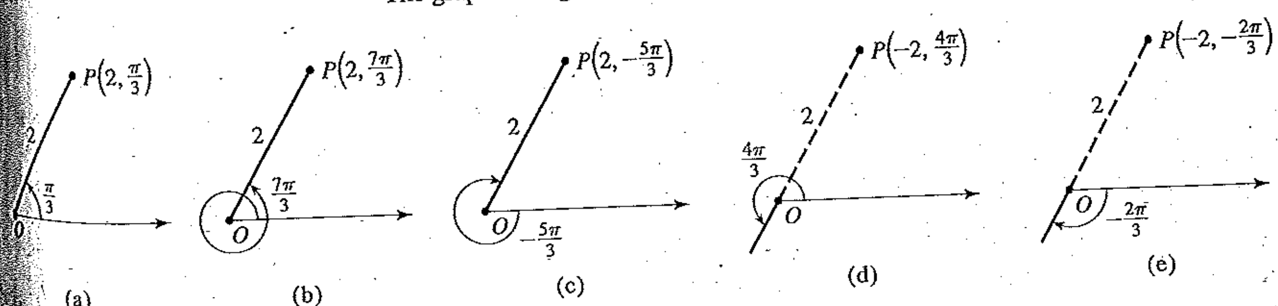


Figure 5

Relationship between Polar and Rectangular Coordinates

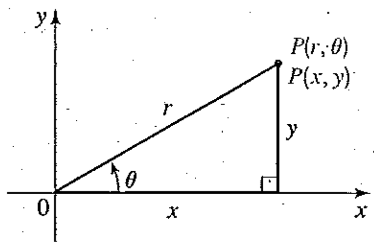


Figure 6

Situations often arise in which we need to consider polar and rectangular coordinates simultaneously. The connection between the two systems is illustrated in Figure 6, where the polar axis coincides with the positive x-axis. The formulas in the following box are obtained from the figure using the definitions of the trigonometric functions and the Pythagorean Theorem. (Although we have pictured the case where $r > 0$ and θ is acute, the formulas hold for any angle θ and for any value of r.)

Relationship between Polar and Rectangular Coordinates

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Example 3 Converting Polar Coordinates to Rectangular Coordinates

Find rectangular coordinates for the point that has polar coordinates $(4, 2\pi/3)$.

Solution Since $r = 4$ and $\theta = 2\pi/3$, we have

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = 4\cdot\left(-\frac{1}{2}\right) = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Thus, the point has rectangular coordinates $(-2, 2\sqrt{3})$. ∎

Example 4 Converting Rectangular Coordinates to Polar Coordinates

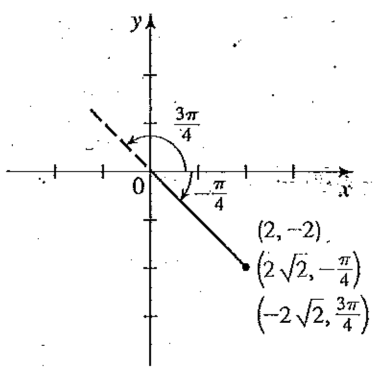


Figure 7

Find polar coordinates for the point that has rectangular coordinates $(2, -2)$.

Solution Using $x = 2$, $y = -2$, we get

$$r^2 = x^2 + y^2 = 2^2 + (-2)^2 = 8$$

so $r = 2\sqrt{2}$ or $-2\sqrt{2}$. Also

$$\tan\theta = \frac{y}{x} = \frac{-2}{2} = -1$$

so $\theta = 3\pi/4$ or $-\pi/4$. Since the point $(2, -2)$ lies in quadrant IV (see Figure 7), we can represent it in polar coordinates as $(2\sqrt{2}, -\pi/4)$ or $(-2\sqrt{2}, 3\pi/4)$. ∎

Note that the equations relating polar and rectangular coordinates do not uniquely determine r or θ. When we use these equations to find the polar coordinates of a point, we must be careful that the values we choose for r and θ give us a point in the correct quadrant, as we saw in Example 4.

Polar Equations

In Examples 3 and 4 we converted points from one coordinate system to the other. Now we consider the same problem for equations.

Example 5 Converting an Equation from Rectangular to Polar Coordinates

Express the equation $x^2 = 4y$ in polar coordinates.

Solution We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 = 4y \qquad \text{Rectangular equation}$$

$$(r\cos\theta)^2 = 4(r\sin\theta) \qquad \text{Substitute } x = r\cos\theta,\ y = r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta \qquad \text{Expand}$$

$$r = 4\frac{\sin\theta}{\cos^2\theta} \qquad \text{Divide by } r\cos^2\theta$$

$$r = 4\sec\theta\tan\theta \qquad \text{Simplify}$$

■

As Example 5 shows, converting from rectangular to polar coordinates is straightforward—just replace x by $r\cos\theta$ and y by $r\sin\theta$, and then simplify. But converting polar equations to rectangular form often requires more thought.

Example 6 Converting Equations from Polar to Rectangular Coordinates

Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$ (c) $r = 2 + 2\cos\theta$

Solution

(a) Since $\sec\theta = 1/\cos\theta$, we multiply both sides by $\cos\theta$.

$$r = 5\sec\theta$$

$$r\cos\theta = 5 \qquad \text{Multiply by } \cos\theta$$

$$x = 5 \qquad \text{Substitute } x = r\cos\theta$$

The graph of $x = 5$ is the vertical line in Figure 8.

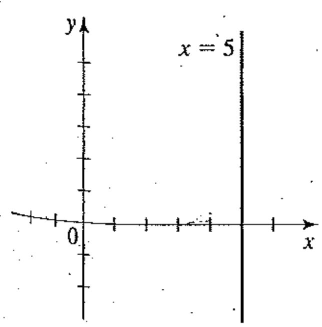


Figure 8

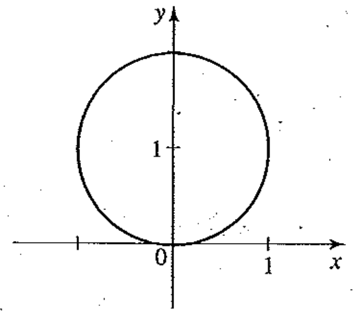


Figure 9

(b) We multiply both sides of the equation by r, because then we can use the formulas $r^2 = x^2 + y^2$ and $r \sin\theta = y$.

$$r^2 = 2r\sin\theta \qquad \text{Multiply by } r$$
$$x^2 + y^2 = 2y \qquad r^2 = x^2 + y^2 \text{ and } r\sin\theta = y$$
$$x^2 + y^2 - 2y = 0 \qquad \text{Subtract } 2y$$
$$x^2 + (y-1)^2 = 1 \qquad \text{Complete the square in } y$$

This is the equation of a circle of radius 1 centered at the point $(0, 1)$. It is graphed in Figure 9.

(c) We first multiply both sides of the equation by r:

$$r^2 = 2r + 2r\cos\theta$$

Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we can convert two of the three terms in the equation into rectangular coordinates, but eliminating the remaining r requires more work:

$$x^2 + y^2 = 2r + 2x \qquad r^2 = x^2 + y^2 \text{ and } r\cos\theta = x$$
$$x^2 + y^2 - 2x = 2r \qquad \text{Subtract } 2x$$
$$(x^2 + y^2 - 2x)^2 = 4r^2 \qquad \text{Square both sides}$$
$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \qquad r^2 = x^2 + y^2$$

In this case, the rectangular equation looks more complicated than the polar equation. Although we cannot easily determine the graph of the equation from its rectangular form, we will see in the next section how to graph it using the polar equation. ∎

8.1 Exercises

1–6 ∎ Plot the point that has the given polar coordinates.

1. $(4, \pi/4)$ **2.** $(1, 0)$ **3.** $(6, -7\pi/6)$

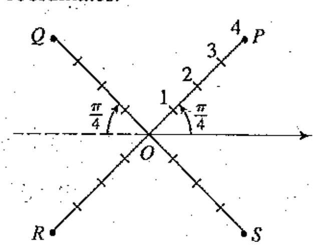
4. $(3, -2\pi/3)$ **5.** $(-2, 4\pi/3)$ **6.** $(-5, -17\pi/6)$

7–12 ∎ Plot the point that has the given polar coordinates. Then give two other polar coordinate representations of the point, one with $r < 0$ and the other with $r > 0$.

7. $(3, \pi/2)$ **8.** $(2, 3\pi/4)$ **9.** $(-1, 7\pi/6)$

10. $(-2, -\pi/3)$ **11.** $(-5, 0)$ **12.** $(3, 1)$

13–20 ∎ Determine which point in the figure, P, Q, R, or S, has the given polar coordinates.



13. $(4, 3\pi/4)$ **14.** $(4, -3\pi/4)$

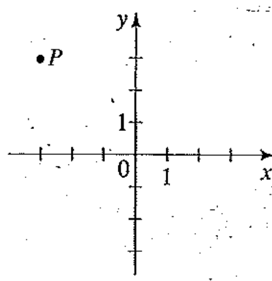
15. $(-4, -\pi/4)$ **16.** $(-4, 13\pi/4)$

17. $(4, -23\pi/4)$ **18.** $(-4, 23\pi/4)$

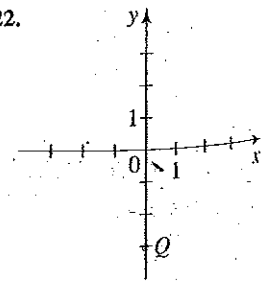
19. $(-4, 101\pi/4)$ **20.** $(4, 103\pi/4)$

21–22 ∎ A point is graphed in rectangular form. Find polar coordinates for the point, with $r > 0$ and $0 < \theta < 2\pi$.

21.

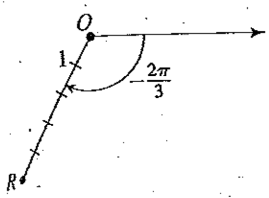


22.

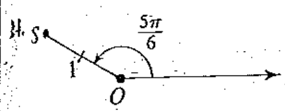


23–24 ■ A point is graphed in polar form. Find its rectangular coordinates.

23.



24.



25–32 ■ Find the rectangular coordinates for the point whose polar coordinates are given.

25. $(4, \pi/6)$
26. $(6, 2\pi/3)$
27. $(\sqrt{2}, -\pi/4)$
28. $(-1, 5\pi/2)$
29. $(5, 5\pi)$
30. $(0, 13\pi)$
31. $(6\sqrt{2}, 11\pi/6)$
32. $(\sqrt{3}, -5\pi/3)$

33–40 ■ Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$.

33. $(-1, 1)$
34. $(3\sqrt{3}, -3)$
35. $(\sqrt{8}, \sqrt{8})$
36. $(-\sqrt{6}, -\sqrt{2})$
37. $(3, 4)$
38. $(1, -2)$
39. $(-6, 0)$
40. $(0, -\sqrt{3})$

41–46 ■ Convert the equation to polar form.

41. $x = y$
42. $x^2 + y^2 = 9$
43. $y = x^2$
44. $y = 5$
45. $x = 4$
46. $x^2 - y^2 = 1$

47–60 ■ Convert the polar equation to rectangular coordinates.

47. $r = 7$
48. $\theta = \pi$
49. $r \cos \theta = 6$
50. $r = 6 \cos \theta$
51. $r^2 = \tan \theta$
52. $r^2 = \sin 2\theta$
53. $r = \dfrac{1}{\sin \theta - \cos \theta}$
54. $r = \dfrac{1}{1 + \sin \theta}$
55. $r = 1 + \cos \theta$
56. $r = \dfrac{4}{1 + 2 \sin \theta}$
57. $r = 2 \sec \theta$
58. $r = 2 - \cos \theta$
59. $\sec \theta = 2$
60. $\cos 2\theta = 1$

Discovery • Discussion

61. The Distance Formula in Polar Coordinates

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(1, 7\pi/6)$, using the formula from part (a).

(c) Now convert the points in part (b) to rectangular coordinates. Find the distance between them using the usual Distance Formula. Do you get the same answer?

8.2 Graphs of Polar Equations

The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations rather than rectangular equations.

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a) on the next page). To plot points in polar coordinates, it is conven-