Math 12 – Precalculus – Chapter 11 test – fall '09 Name____________________

Show your work for credit. Write all responses on separate paper.

In 1 – 2, verify the identity.

1. $\frac{1}{1-\cos\theta}-\frac{1}{1+\cos\theta}=2\csc\theta\cot\theta$

2. $\sin x+\sqrt{3}\cos x=2\sin\left(x-\frac{\pi}{6}\right)$

In 3 – 5, solve the equation in the interval $[0,2\pi)$

3. $\sqrt{2}\cos^2 x-\cos x=0$

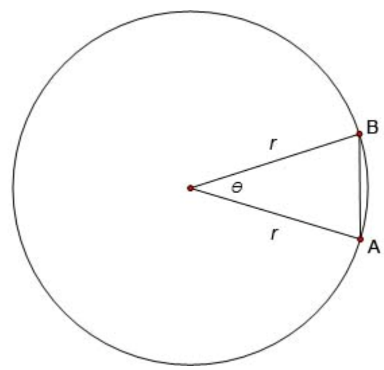
4. $3-2\cos^2 t=3\sin t$

5. $\cos 3x-2\sin x=0$ *Hint*: start by expanding cos(2x + x)

6. Show how to use the half angle formula to find $\sin\frac{13\pi}{12}$.

7. Find $\cos(2\theta)$, $\sin(2\theta)$ and $\tan(2\theta)$ given that $\cos(\theta)=-\frac{4}{5}$ and x is in quadrant III.

8. Suppose the chord AB in a circle of radius r subtends central angle θ, as shown in the diagram at right. Show that the length of AB is $2r\sin\frac{\theta}{2}$.



9. Find the exact value of $\sin\left(\tan^{-1}\left(-\frac{24}{7}\right)\right)$ and write it in simplest radical form.

10. Factor $8\sin^4 x-10\sin^2 x+3$ and find all solutions to the equation $8\sin^4 x-10\sin^2 x+3=0$.

11. Use the sum to product identity to find all solutions to $\cos x+\sin 3x=0$ (do not approximate.)

12. Find all solutions to $\sin^3 4x-2\cos 4x=0$.

Math 12 – Precalculus – Chapter 11 test Solutions.

In 1 – 2, verify the identity.

1. $\dfrac{1}{1-\cos\theta}-\dfrac{1}{1+\cos\theta}=2\csc\theta\cot\theta$

SOLN: (in the form of a sequence of equivalent equations, the last of which is an obvious identity)

$$\frac{1}{1-\cos\theta}-\frac{1}{1+\cos\theta}=2\csc\theta\cot\theta$$
$$\frac{1+\cos\theta}{1-\cos^2\theta}-\frac{1-\cos\theta}{1-\cos^2\theta}=2\frac{\cos\theta}{\sin^2\theta}$$
$$\frac{2\cos\theta}{1-\cos^2\theta}=\frac{2\cos\theta}{1-\cos^2\theta}$$

2. $\sin x+\sqrt{3}\cos x=2\sin\left(x-\dfrac{\pi}{6}\right)$t

SOLN: Well, this is embarrassing. Start with the RHS and you have

$$2\sin\left(x-\frac{\pi}{6}\right)=2\left(\cos\frac{\pi}{6}\sin x-\sin\frac{\pi}{6}\cos x\right)=\sqrt{3}\sin x-\cos x$$
$$\neq\sin x+\sqrt{3}\cos x=2\left(\cos\frac{\pi}{3}\sin x+\sin\frac{\pi}{3}\cos x\right)=2\sin\left(x+\frac{\pi}{3}\right)$$

In 3 – 5, solve the equation in the interval $[0,2\pi)$

3. $\sqrt{2}\cos^2 x-\cos x=0$

SOLN: $\sqrt{2}\cos^2 x-\cos x=\cos x\left(\sqrt{2}\cos x-1\right)=0\Leftrightarrow\cos x=0$ or $\cos x=\dfrac{\sqrt{2}}{2}$

Thus the solutions are $x\in\left\{\dfrac{\pi}{4},\dfrac{\pi}{2},\dfrac{3\pi}{2},\dfrac{7\pi}{4}\right\}$

4. $3-2\cos^2 t=3\sin t$

SOLN:

$$3-2\cos^2 t=3\sin t\Leftrightarrow 3-2\left(1-\sin^2 t\right)=3\sin t\Leftrightarrow 2\sin^2 t-3\sin t+1=0\Leftrightarrow\left(2\sin t-1\right)\left(\sin t-1\right)=0$$

Which is equivalent to either $\sin t=\dfrac{1}{2}$ or $\sin t=1$ so the solutions are $t\in\left\{\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6}\right\}$

5. $\cos 3x-2\sin x=0$ *Hint*: start by expanding $\cos(2x+x)$

SOLN: (full disclosure: I had meant to have $\sin 3x-2\sin x=0$ which would lead to the following

$$\sin 3x-2\sin x=0\Leftrightarrow\sin x\cos 2x+\sin 2x\cos x-2\sin x=(1-2\sin^2 x)\sin x+2\sin x\cos^2 x=0$$

So, factoring out sin$x$ and using the Pythagorean identity:

$(2\cos^2 x+1-2\sin^2 x)\sin x=0\Leftrightarrow\left(4\cos^2 x-1\right)\sin x=0$ which is true iff sin$x$ = 0 or $\cos x=\pm\dfrac{1}{2}$ so the solution set is $x\in\left\{0,\dfrac{\pi}{3},\dfrac{2\pi}{3},\pi\right\}$ Nice and simple, huh? Here's the solution to the problem posed:

$$\cos 3x - 2\sin x = \cos(2x + x) - 2\sin x = \cos 2x \cos x - \sin 2x \sin x - 2\sin x$$

$$\Leftrightarrow \left(2\cos^2 x - 1\right)\cos x - 2\sin^2 x \cos x = 2\sin x$$

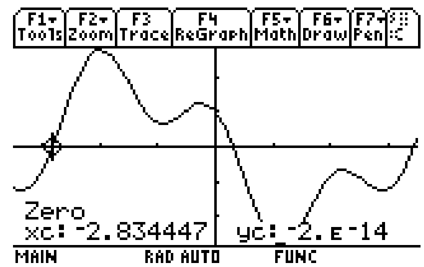
$$\Leftrightarrow 2\cos^3 x - \cos x - 2\left(1 - \cos^2 x\right)\cos x = 2\sin x$$

$$\Leftrightarrow 4\cos^3 x - 3\cos x = 2\sin x$$

$$\Rightarrow \left(4\cos^3 x - 3\cos x\right)^2 = 4\sin^2 x$$

$$\Leftrightarrow 16\cos^6 x - 24\cos^4 x + 13\cos^2 x - 4 = 0$$

$$\Leftrightarrow p(u) = 16u^3 - 24u^2 + 13u - 4 = 0$$



6. This last polynomial has no negative roots (Descartes' rule of signs) and 3 or 1 positive roots. Now $p(0) = -4$ and $p(u) = (u - 1)(16u^2 - 8u + 5) + 1$ so $p(1) = 1$ indicates (intermediate value theorem) there's a zero between $u = 0$ and $u = 1$, probably closer to 1.
$p(u) = (u - 0.9)(16u^2 - 9.6u + 4.36) - 0.076$ so let's take 0.91 as an approximation to the zero there. Then $u = \cos^2 x \approx 0.91 \Leftrightarrow \cos x \approx \pm 0.95 \Leftrightarrow \boxed{x \approx \pm \arccos(0.95) + k\pi,\ k \in \mathbb{Z}}$. But, in solving the equation, we equated square, which can bring in an extraneous solution. At this stage, work become much easier with a calculator. For instance, arcos(0.95) is near 0.32, is not easily computed by hand at this stage. $\pi - 0.32$ is approximately -2.83 and is also a solution. So the solutions seem to be good, as the graph shows.

7. Show how to use the half angle formula to find $\sin\frac{13\pi}{12}$.

SOLN: $\sin\frac{13\pi}{12} = -\sqrt{\frac{1 - \cos(13\pi/6)}{2}} = -\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = -\frac{\sqrt{2 - \sqrt{3}}}{2}$

8. Find $\cos(2\theta)$, $\sin(2\theta)$ and $\tan(2\theta)$ given that $\cos(\theta) = -\frac{4}{5}$ and θ is in quadrant III.

SOLN: $\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{16}{25}\right) - 1 = \frac{7}{25}$

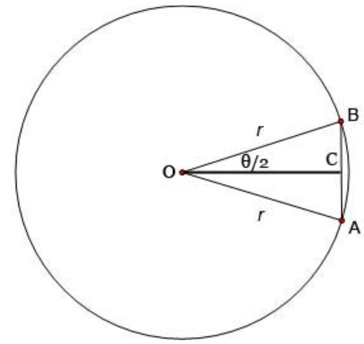
Since $\pi \le \theta \le 3\pi/2$, $2\pi \le 2\theta \le 3\pi$ where sine is positive.

Thus $\sin(2\theta) = \sqrt{1 - \left(\frac{7}{25}\right)^2} = \sqrt{1 - \frac{49}{625}} = \sqrt{\frac{576}{625}} = \frac{24}{25}$

The tangent function is then easy to figure as $\sin 2\theta / \cos 2\theta = 24/7$.

9. Suppose the chord AB in a circle of radius r subtends central angle θ, as shown in the diagram at right. Show that the length of AB is $2r\sin\frac{\theta}{2}$.

SOLN: Construct perpendicular bisector of AB and observe that this will pass through the circle center. Then sin(θ/2) = BC/r meaning that $AB = 2BC = 2r\sin(\theta/2)$.



10. Find the exact value of $\sin\left(\tan^{-1}\left(-\frac{24}{7}\right)\right)$ and write it in simplest radical form.

SOLN: $\tan^{-1}\left(-\frac{24}{7}\right) = -\tan^{-1}\left(\frac{24}{7}\right)$. Recall that 7,24,25 is a Pythagorean triple, so

$$\sin\left(\tan^{-1}\left(-\frac{24}{7}\right)\right) = -\frac{24}{25}$$

11. Factor $8\sin^4 x - 10\sin^2 x + 3$ and find all solutions to the equation $8\sin^4 x - 10\sin^2 x + 3 = 0$.

SOLN:

$$8\sin^4 x - 10\sin^2 x + 3 = 8\sin^4 x - 6\sin^2 x - 4\sin^2 x + 3 = 2\sin^2 x\left(4\sin^2 x - 3\right) - 1\left(4\sin^2 x - 3\right)$$

$$= \left(2\sin^2 x - 1\right)\left(4\sin^2 x - 3\right) = 0 \Leftrightarrow \sin x = \pm\frac{\sqrt{2}}{2} \text{ or } \sin x = \pm\frac{\sqrt{3}}{2}$$

Thus all solutions are of the form $\frac{\pi}{4} + \frac{k\pi}{2}$, $\frac{\pi}{3} + k\pi$ or $\frac{2\pi}{3} + k\pi$

12. Use the sum to product identity to find all solutions to $\cos x + \sin 3x = 0$ (do not approximate.)

SOLN:

$$\cos x + \sin 3x = \sin\left(\frac{\pi}{2} - x\right) + \sin 3x = 2\sin\left(\frac{\frac{\pi}{2} - x + 3x}{2}\right)\cos\left(\frac{\frac{\pi}{2} - x - 3x}{2}\right)$$

$$= 2\sin\left(\frac{\pi}{4} + x\right)\cos\left(\frac{\pi}{4} - 2x\right) = 0$$

So either $\sin\left(\frac{\pi}{4} + x\right) = 0 \Leftrightarrow \frac{\pi}{4} + x = k\pi \Leftrightarrow \boxed{x = -\frac{\pi}{4} + k\pi}$ or

$$\cos\left(\frac{\pi}{4} - 2x\right) = 0 \Leftrightarrow \frac{\pi}{4} - 2x = \frac{(2k+1)\pi}{2} \Leftrightarrow \boxed{x = \frac{\pi}{8} - \frac{(2k+1)\pi}{4}}$$

13. Find all solutions to $\sin^3 4x - 2\cos 4x = 0$.

SOLN: (full disclosure: I had meant this to be $\sin^3 4x - 2\sin 4x = 0$)

$$\sin^3 4x - 2\cos 4x = 0 \Rightarrow \sin^6 4x = \left(2\cos 4x\right)^2 \Leftrightarrow \sin^6 4x = 4\left(1 - \sin^2 4x\right) \Leftrightarrow \sin^6 4x + 4\sin^2 4x - 4 = 0$$

Let $u = \sin^2 4x$ so the equation is cubic in u: $f(u) = u^3 + 4u - 4 = 0$. Now the possible rational zeros for this cubic are in the set ±{1,2,4} and since $f(0)f(1) < 0$, by the Intermediate Value theorem there is a zero between $u = 0$ and $u = 1$. $f(u) = (u - 1)(u^2 + u + 5) + 1$ so 1 is an upper bound on the zeros of f. Thus f has at least one irrational zero in (0,1). We can use synthetic division to approximate a decimal approximation for the zero of the cubic. As shown at right, the best you can reasonably

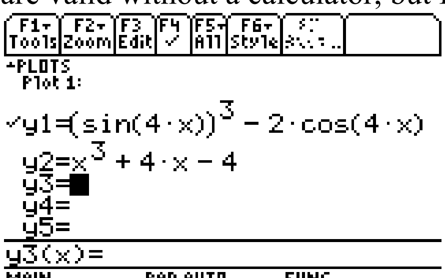
0	1	0	4	−4
1	1	1	5	1
0.9	1	0.9	4.81	0.329
0.8	1	0.8	4.64	−0.288
0.85	1	0.85	4.7225	0.014125

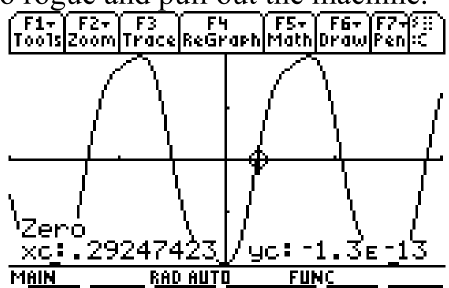
expect to approximate the zero is u near 0.85, which means $\sin^2 4x \approx 0.85 \Leftrightarrow \sin 4x \approx \pm 0.92$ and so $4x$ will be any angle coterminal with $\pm\arcsin(0.92)$ or $\pi \pm \arcsin(0.92)$, that is

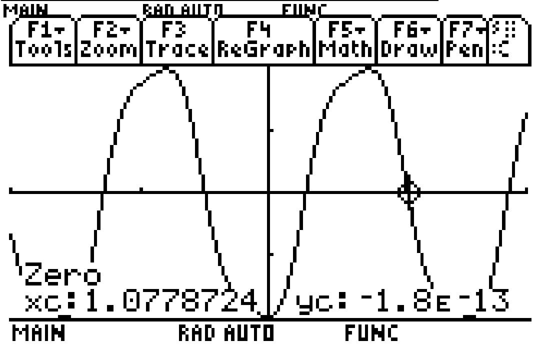
$4x = \pm\arcsin(0.92) + 2k\pi$ or $4x = \pi \pm \arcsin(0.92) + 2k\pi$, and solving for x we get

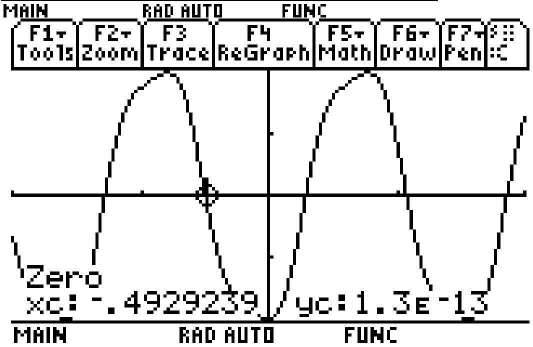
$$\boxed{x = \frac{\pm\arcsin(0.92)}{4} + \frac{k\pi}{2} \text{ or } x = \frac{\pi \pm \arcsin(0.92)}{4} + \frac{k\pi}{2}}$$

However, arriving at these solutions involved equating the square of the LHS with the square of the RHS, which can lead to extraneous solutions. There's probably a clever way to check which solutions are valid without a calculator, but I'm going to go rogue and pull out the machine.









Now arcsin(0.92)/4 is approximately 0.292 and qualifies as a solution, but evidently its negation does not. However, arcsin(0.92)/4 – π/4 is approximately -0.493, so it looks like the solutions are near either 0.292 plus any integer multiple of π/2 or -0.493 plus any integer multiple of π/2.