Math 12 – Precalculus – Final Exam – Fall ‘09 Name_______________________________

Write all responses on separate paper. Show your work for credit. Do not use a calculator.

1. Consider the point with polar coordinates, $(r,\theta)=\left(4,\frac{\pi}{4}\right)$
   a. Plot the point in the polar coordinate plane.
   b. Find the rectangular coordinates of the point.

2. Consider the polar function $r = 4 + 8\sin\theta$.
   a. Tabulate values of r for $\theta = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.
   b. Use symmetry and values in your table to sketch a graph for the function.

3. Consider the complex number $z = \frac{\sqrt{3}}{2} - \frac{i}{2}$
   a. Plot the point in the complex plane and give the polar form of z.
   b. Use Demoivre’s formula to compute z^2 and z^3 and plot these together with z in the polar plane.

4. Find the slope-intercept form ($y = mx + b$) of the equations for the two lines intersecting the circle $x^2 + y^2 = 1$ where $x = -1$ and where $x = ½$. Note there are two points on the circle where $x = ½$.

5. Consider the polynomial function $f(x) = x^3\left(x^2-4\right)^2$
   a. State the zeros of the function and the multiplicity of each zero.
   b. Sketch a graph showing how the function behaves near its zeros and the end behavior.

6. Consider the polynomial function $p(x) = 3x^4 + 8x^3 - 24x^2 - 73x - 24$
   a. State why the polynomial satisfies the condition of the rational zeros theorem and list all possible rational zeros indicated by the conclusion of that theorem.
   b. Use synthetic division to show that $x = -\frac{8}{3}$ is a rational zero and then use the factor theorem to write the polynomial as the product of a linear and a cubic factor.
   c. Use the fact that $x = 3$ is also a zero to find the two irrational zeros of $p(x)$ and write these in simplified radical form.

7. Suppose $R(x) = \frac{f(x)}{x^3-8}$ is a rational function (that is, f and g are polynomial functions) .
   a. Find a function f of degree 3 with integer coefficients so that R has x-intercepts at $x = \frac{1}{2}$ and $x = \sqrt{2}$
   b. Write an equation for the vertical asymptote(s) of R. *Hint*: This is in the form, x = a constant.
   c. Given the function f you’ve got, write an equation for the horizontal asymptote of R.

8. Evaluate each logarithmic expression.
   a. $\log_5\sqrt{125}$ b. $\log_3 135 - \log_3 5$ c. $\log_6 4 + \log_6 9$

9. The initial population of rats in a lab experiment is 100.
After 8 days, the population has grown to 430.
   a. Find an exponential function that models the rat population after t weeks.
   b. Find the population after 4 days. Approximate as best you can without a calculator.
   c. When will the population reach 1728? Again, approximate as best you can.
   d. Sketch a graph of the population function.

10. Consider the logarithmic function $f(x) = \log_3(2x+3)$
   a. Find the x- and y-intercepts for this function.
   b. Find a formula for the inverse function.
   c. Graph the function and its inverse together, showing the symmetry through the line $y = x$

11. Solve the equation:
   a. $\log_2(4x+4) + \log_2(x+1) = 10$
   b. $10 - e^{1-x^2} = 8$

12. Suppose a point in the fourth quadrant of the unit circle has x coordinate $\frac{1}{5}$.
   a. Draw the unit circle in the Cartesian plane showing the position of this point.
   b. Find the radian measure of the smallest positive polar angle θ that terminates at this point.
   c. What are the coordinates of the terminal point for $\theta - \pi$?
   d. What are the coordinates of the terminal point for $\theta - \frac{\pi}{2}$?

13. Find all x-intercepts of $f(x) = 1 - \sqrt{2}\sin\left(2x - \frac{\pi}{3}\right)$. Write the values in exact form.

14. If $\tan\theta = -\frac{1}{4}$ and $\sin\theta < 0$, find $\cos\theta$ and $\cos\frac{\theta}{2}$.

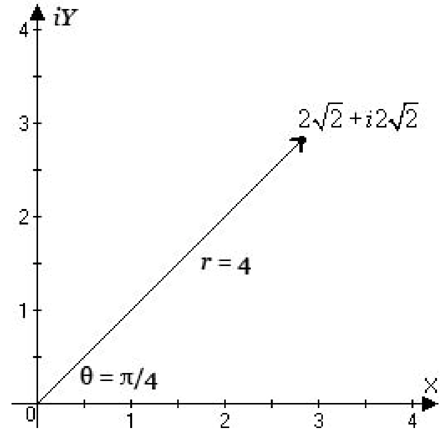
15. Express $\sec\left(\arctan\frac{1}{2}\right)$ in simplest radical form.

16. Find all solutions to the equation: $\cos 2x + \sin x = 1$ in exact form.

17. For the angles $\alpha = \arctan(1/2)$ and $\beta = \arctan(4/3)$ find exact value for
   a. $\sin(\alpha + \beta)$
   b. $\sin(2\alpha + \beta)$

Math 12 – Precalculus – Final Exam Solutions – Fall '09

1. Consider the point with polar coordinates,
   $(r,\theta)=\left(4,\frac{\pi}{4}\right)$



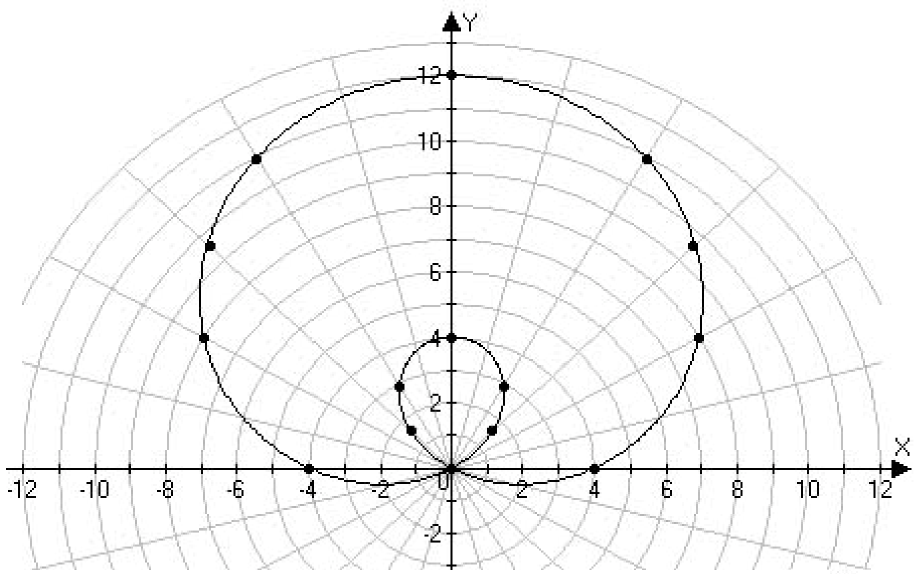
   a. Plot the point in the polar coordinate plane.

   b. Find the rectangular coordinates of the point.
      SOLN: $2\sqrt{2}+i2\sqrt{2}=\left(2\sqrt{2},2\sqrt{2}\right)$

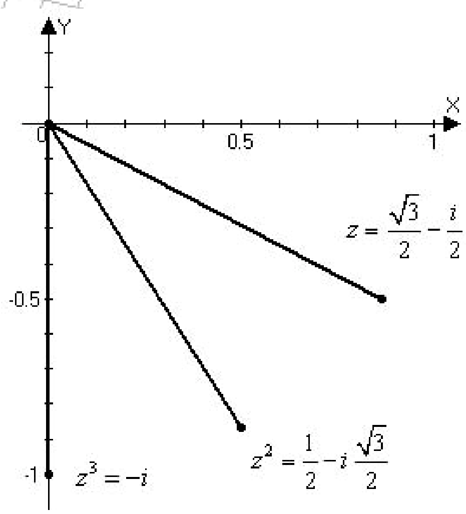
2. Consider the polar function $r=4+8\sin\theta$.
   a. Tabulate values of r for $\theta = 0$, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$.

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
r	-4	$4-4\sqrt{3}$ ≈ -2.9	$4-4\sqrt{2}$ ≈ -1.6	0	4	8	$4+4\sqrt{2}$ ≈ 9.6	$4+4\sqrt{3}$ ≈ 10.9	12

b. Use symmetry and values in your table to sketch a graph for the function.



3. Consider the complex number $z=\frac{\sqrt{3}}{2}-\frac{i}{2}$

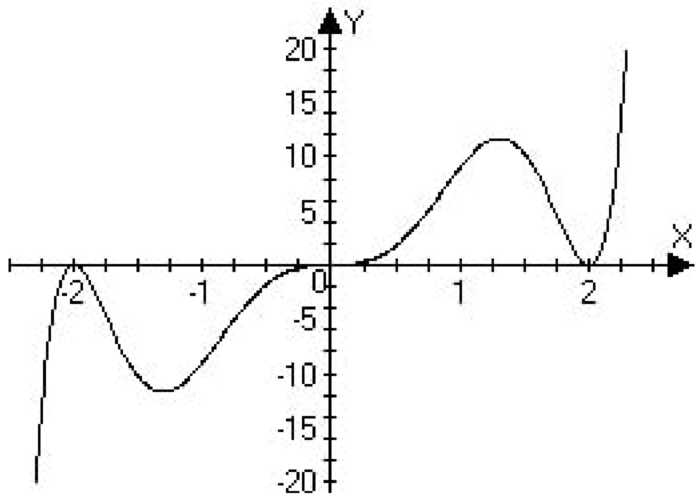


   a. Plot z in the complex plane and give the polar form of z.
      SOLN: $\theta = \arctan(-1/\sqrt{3}) = -\pi/6$ whereas
      $r=\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=1$ so that $(r,\theta) = (1, -\pi/6)$

   b. Use Demoivre's formula to compute z^2 and z^3 and plot these: $z^2=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, and
      $z^3=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)=-i$

4. Find the slope-intercept form ($y = mx + b$) of the equations for the two lines intersecting the circle $x^2 + y^2 = 1$ where $x = -1$ and where $x = ½$. Note there are two points on the circle where $x = ½$.
SOLN: There is only one point on the circle where $x = -1$, that is $(-1,0)$. There are two points where $x = ½$: $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ so the slopes are $m = \dfrac{0 \pm \frac{\sqrt{3}}{2}}{-1-\frac{1}{2}} = \pm\frac{\sqrt{3}}{3}$ and the equations for the lines are $y = \pm\frac{\sqrt{3}}{3}(x+1)$

5. Consider the polynomial function $f(x) = x^3\left(x^2-4\right)^2$
   a. State the zeros of the function and the multiplicity of each zero.
      $x = 0$ is a zero of multiplicity 3 and $x = \pm 2$ are zeros of multiplicity 2.
   b. Sketch a graph showing how the function behaves near its zeros and the end behavior.
      SOLN: At right.



6. Consider the polynomial function $p(x) = 3x^4 + 8x^3 - 24x^2 - 73x - 24$
   a. State why the polynomial satisfies the condition of the rational zeros theorem and list all possible rational zeros indicated by the conclusion of that theorem.
      SOLN: The coefficients are all integers. Possible rational zeros are in $\pm\left\{1,2,3,4,6,8,12,24,\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{8}{3}\right\}$
   b. Use synthetic division to show that $x = -\frac{8}{3}$ is a rational zero and then use the factor theorem to write the polynomial as the product of a linear and a cubic factor.
      SOLN:
$$\begin{array}{r|rrrr|r} & 3 & 8 & -24 & -73 & -24 \\ \hline -\frac{8}{3} & 3 & 0 & -24 & -9 & 0 \end{array}$$
      $p(x) = (3x+8)\left(x^3-8x-3\right)$
   c. Use the fact that $x = 3$ is also a zero to find the two irrational zeros of $p(x)$ and write these in simplified radical form.
      SOLN: $p(x) = (3x+8)(x-3)\left(x^2+3x+1\right)$ so the irrational zeros are $\frac{-3\pm\sqrt{5}}{2}$

7. Suppose $R(x)=\dfrac{f(x)}{x^3-8}$ is a rational function (that is, *f* and *g* are polynomial functions) .

   a. Find a function f of degree 3 with integer coefficients so that *R* has x-intercepts at $x=\dfrac{1}{2}$ and $x=\sqrt{2}$

      SOLN: $f(x)=(2x-1)(x^2-2)=2x^3-x^2-4x+2$

   b. Write an equation for the vertical asymptote(s) of *R*. *Hint*: This is in the form, x = a constant.
      SOLN $x = 2$ is the vertical asymptote.

   c. Given the function *f* you've got, write an equation for the horizontal asymptote of *R*.
      SOLN: $y = 2$ is the horizontal asymptote.

8. Evaluate each logarithmic expression.

   a. $\log_5\sqrt{125}=\dfrac{3}{2}$ b. $\log_3 135-\log_3 5=3$ c. $\log_6 4+\log_6 9=2$

9. The initial population of rats in a lab experiment is 100.
   After 8 days, the population has grown to 430.

   a. Find an exponential function that models the rat population after *t* weeks.

      SOLN: 8 days is 8/7 of a week so $P(t)=100(4.3)^{7t/8}$

      will meet the condition that $P\left(\dfrac{8}{7}\right)=100(4.3)^1=430$

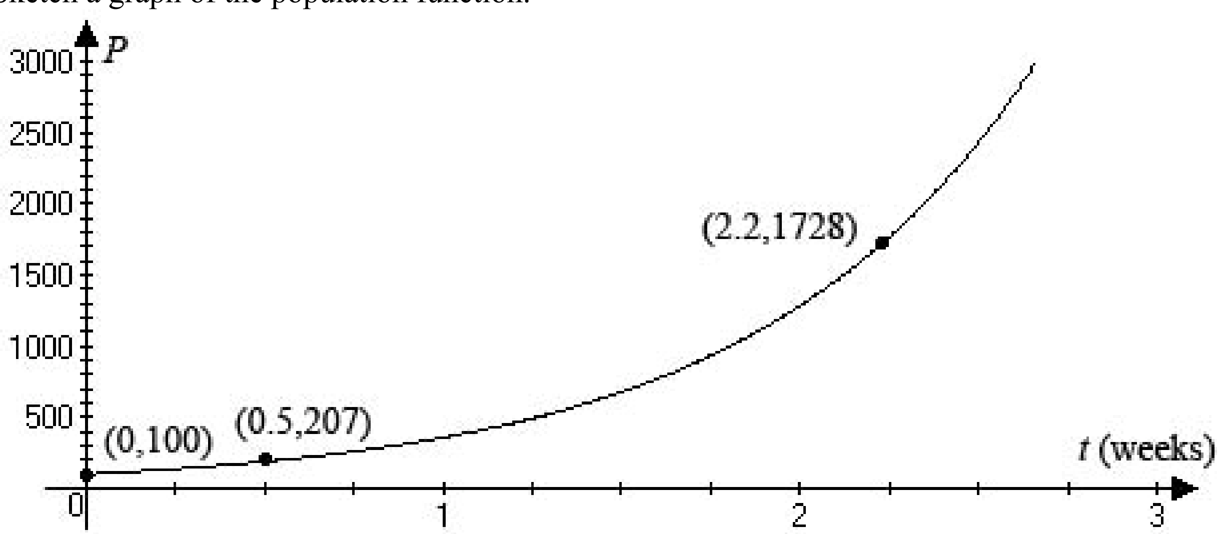
   b. Find the population after 4 days. Approximate as best you can without a calculator.

      SOLN: $P\left(\dfrac{4}{7}\right)=100(4.3)^{1/2}\approx 100(2.07)=207$

   c. When will the population reach 1728? Again, approximate as best you can.

      SOLN: $P(t)=100(4.3)^{7t/8}=1728 \Leftrightarrow (4.3)^{7t/8}=17.28 \Leftrightarrow \dfrac{7t}{8}=\dfrac{\log 17.28}{\log 4.3}\approx 2$ so $t\approx 2.2$

   d. Sketch a graph of the population function.



10. Consider the logarithmic function $f(x) = \log_3(2x+3)$

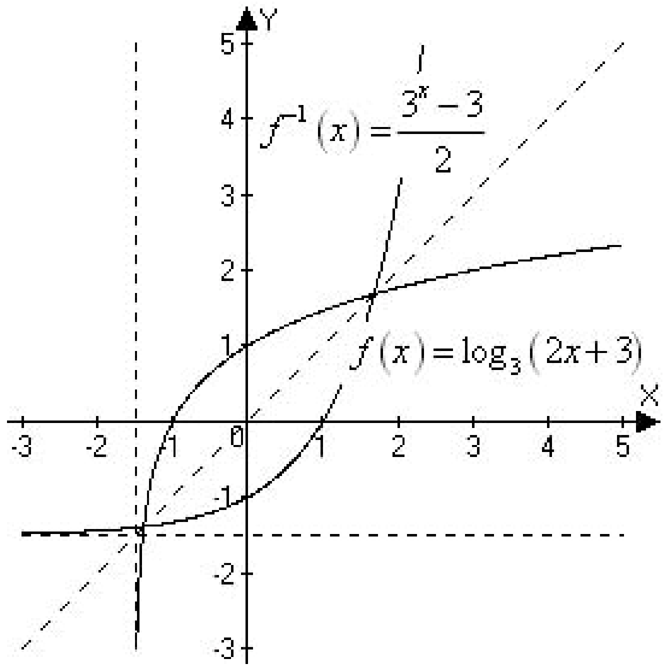
a. Find the x- and y-intercepts for this function.
SOLN: (0,1) and (–1,0)

b. Find a formula for the inverse function.

SOLN: $f^{-1}(x) = \dfrac{3^x - 3}{2}$

c. Graph the function and its inverse together, showing the symmetry through the line $y = x$
SOLN: A table of values for f can be simply reversed (swap x and y) to get a table for f^{-1}

x	$f(x)$
$-\frac{4}{3}$	-1
-1	0
0	1
3	2



11. Solve the equation:

a. $\log_2(4x+4) + \log_2(x+1) = 10$

SOLN $\log_2(4x+4)(x+1) = 10 \Leftarrow 4(x+1)^2 = 2^{10} \Leftrightarrow (x+1)^2 = 256 \Leftrightarrow x+1 = \pm 16$

$\Leftrightarrow \boxed{x = 15}$ or $x = -17$

x has got to be bigger than –1 .

b. $10 - e^{1-x^2} = 8$

SOLN: $e^{1-x^2} = 2 \Leftrightarrow 1 - x^2 = \ln 2 \Leftrightarrow x^2 = 1 - \ln 2 \Leftrightarrow x = \pm\sqrt{1 - \ln 2}$

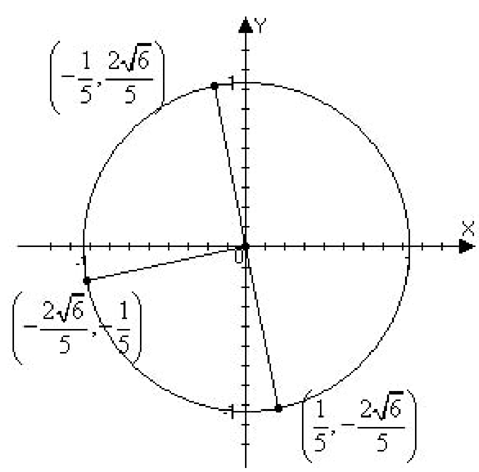
12. Suppose a point in the fourth quadrant of the unit circle has x coordinate $\frac{1}{5}$.

a. Draw the unit circle in the Cartesian plane showing the position of this point SOLN →

b. Find the radian measure of the smallest positive polar angle θ that terminates at this point.
$2\pi - \arctan(2\sqrt{6})$

c. What are the coordinates of the terminal point for $\theta - \pi$? SOLN: $\left(-\frac{1}{5}, \frac{2\sqrt{6}}{5}\right)$

d. What are the coordinates of the terminal point for $\theta - \frac{\pi}{2}$? SOLN: $\left(-\frac{2\sqrt{6}}{5}, -\frac{1}{5}\right)$



13. Find all x-intercepts of $f(x)=1-\sqrt{2}\sin\left(2x-\frac{\pi}{3}\right)$. Write the values in exact form.

SOLN:

$$1-\sqrt{2}\sin\left(2x-\frac{\pi}{3}\right)=0 \Leftrightarrow \sin\left(2x-\frac{\pi}{3}\right)=\frac{\sqrt{2}}{2} \Leftrightarrow 2x-\frac{\pi}{3}=\pm\frac{\pi}{4}+\frac{(4k+1)\pi}{2} \Leftrightarrow \boxed{x=\pm\frac{\pi}{8}+\frac{(12k+5)\pi}{12}}$$

14. If $\tan\theta=-\frac{1}{4}$ and $\sin\theta<0$, find $\cos\theta$ and $\cos\frac{\theta}{2}$.

SOLN: Evidently, θ is in QIV so $\frac{3\pi}{4}\le\frac{\theta}{2}\le\pi$ is in QII and $\frac{3\pi}{8}\le\frac{\theta}{4}\le\frac{\pi}{2}$ is in QI.

Now $x^2+y^2=17y^2=1 \Rightarrow \sin\theta=-\frac{\sqrt{17}}{17}$, $\cos\theta=\frac{4\sqrt{17}}{17}$. Thus

$$\cos\frac{\theta}{2}=-\sqrt{\frac{1+\frac{4\sqrt{17}}{17}}{2}}=-\sqrt{\frac{17+4\sqrt{17}}{34}}=-\frac{\sqrt{578+136\sqrt{17}}}{34}\approx -0.9925$$

15. Express $\cos\left(\arctan\frac{1}{2}\right)$ in simplest radical form.

SOLN: $\cos\left(\arctan\frac{1}{2}\right)=\frac{2\sqrt{5}}{5}$

16. Find all solutions to the equation: $\cos 2x+\sin x=1$ in exact form.

SOLN: $\cos 2x+\sin x=1 \Leftrightarrow 1-2\sin^2 x+\sin x=1 \Leftrightarrow \sin x(2\sin x-1)=0$. Thus

$\sin x=0 \Leftrightarrow \boxed{x=k\pi}$ or $\sin x=\frac{1}{2} \Leftrightarrow \boxed{x=\frac{(4k+1)\pi}{2}\pm\frac{\pi}{3}}$.

17. For the angles $\alpha=\arctan(1/2)$ and $\beta=\arctan(4/3)$ find exact value for

a. $\sin(\alpha+\beta)$

SOLN: $\frac{1}{\sqrt{5}}\frac{3}{5}+\frac{2}{\sqrt{5}}\frac{4}{5}=\frac{11\sqrt{5}}{25}$

b. $\sin(2\alpha+\beta)$

SOLN: $\frac{3}{5}\sin 2\alpha+\frac{4}{5}\cos 2\alpha=\frac{12}{25}+\frac{12}{25}=\frac{24}{25}$