Math 12 – Polynomial and Rational Functions Test – Fall '09 Name___________________________

Show your work for credit. Write all responses on separate paper.

1. Consider the function $f(x) = 3 - (x-1)^3$
   a. Multiply and combine like terms to show that $f(x) = -x^3 + 3x^2 - 3x + 4$

   b. What is the degree of the polynomial? What is the leading coefficient?
   What does this indicate for the end behavior (as $|x| \to \infty$) of the function f?

   c. Simplify $f(-x)$ and use the result to explain what Descartes' rule of signs says about the number of negative zeros of $f(x)$.

   d. List the possible rational zeros and use synthetic division to show that $f(x)$ has no rational zeros.

   e. Use the intermediate value theorem to show that $f(x)$ has a zero between $x = 2$ and $x = 3$.

2. Let $p(x) = x^4 - x^2$
   a. Factor p completely and find all the zeros.

   b. Show that $p(x-1) - 12 = x^4 - 4x^3 + 5x^2 - 2x - 12$.

   c. Complete the tables below (on separate paper):

x	-2	-1	$-1/2$	0	$1/2$	1	2
$p(x)$							

x	-1	0	$1/2$	1	$3/2$	2	3
$p(x-1)-12$							

   d. Sketch graphs for $y = p(x)$ and $y = p(x-1) - 12$ together
   showing how one is a horizontal and vertical shift of the other.

   e. Describe the three roots for $y = p(x-1) - 12$. How many are rational,
   how many are irrational and how many are not even real? How do you know? Explain.

3. Find a polynomial function with degree 3, integer coefficients and zeros at $x = -2$, $x = 3 + \sqrt{2}\,i$.
   Expand to descending powers form.

4. Find a formula for the polynomial of degree 5 which has a root of multiplicity 3 at $x = -1$, a root of multiplicity 2 at $x = 3$ and a y-intercept at (0,3). Sketch a graph showing these key features.

5. Consider $y = \dfrac{2x^2 + x - 1}{x^2 - 4}$.
   a. Factor the numerator and denominator.
   b. What are the intercepts for this function?
   c. What are the vertical asymptotes?
   d. What is the horizontal asymptote?
   e. Plot additional points, as necessary, to get the shape of this function and sketch a graph.

Math 12 – Polynomial and Rational Functions Test Solutions – Fall '09

1. Consider the function $f(x) = 3 - (x-1)^3$
   a. Multiply and combine like terms to show that $f(x) = -x^3 + 3x^2 - 3x + 4$
   SOLN: $f(x) = 3 - (x-1)^3 = 3 - (x^3 - 3x^2 + 3x - 1) = -x^3 + 3x^2 - 3x + 4$

   b. What is the degree of the polynomial? What is the leading coefficient?
   What does this indicate for the end behavior (as $|x| \to \infty$) of the function *f*?
   SOLN: The polynomial has leading coefficient -1 and degree = 3.
   The long term behavior is $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$
   c. Simplify *f*(–*x*) and use the result to explain what Descartes' rule of signs says about the number of negative zeros of *f*(*x*) .
   SOLN: $f(x) = 3 - (x-1)^3 = -(-x)^3 + 3(-x)^2 - 3(-x) + 4 = x^3 + 3x^2 + 3x + 4$
   Since there are no sign changes in the coefficients of *f*(*x*), there are no negative zeros for *f*(*x*).
   d. List the possible rational zeros and use synthetic division to show that *f*(*x*) has no rational zeros.
   SOLN: Since the coefficients of *f*(*x*), the set of possible rational zeros arrived at by the rational zeros theorem is ±{1,2,4}. We don't need to check the negatives (c above) and the values 1, 2 and 4 are not zeros. Therefore there are no rational zeros.

	–1	3	–3	4
1	–1	2	–1	3
2	–1	1	–1	2
4	–1	–1	–7	–24

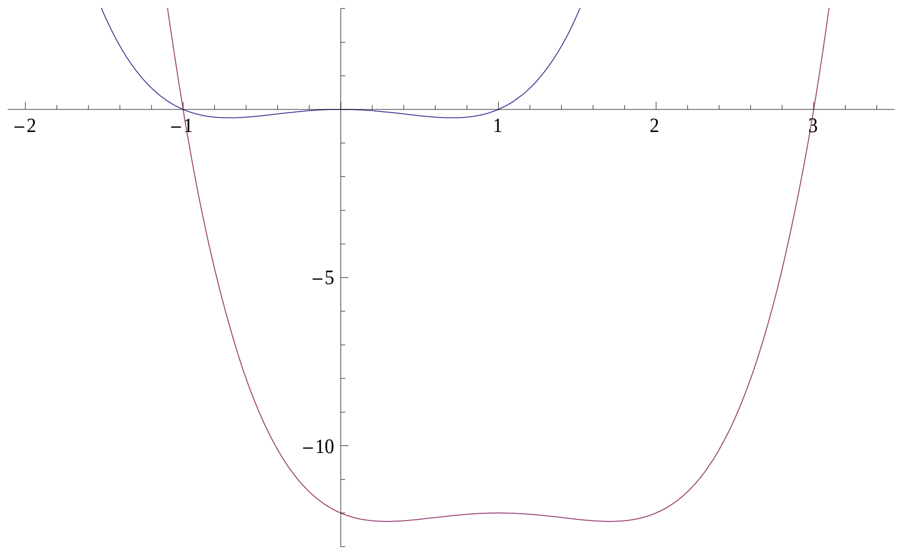
   e. Use the intermediate value theorem to show that *f*(*x*) has a zero between *x* = 2 and *x* = 3.
   SOLN: *f*(2) = 2 and *f*(3) = –5 so *f* (a continuous function) changes sign between 2 and 3 and must, by the intermediate value theorem, have a zero in that interval.

2. Let $p(x) = x^4 - x^2$
   a. Factor *p* completely and find all the zeros.
   SOLN: $p(x) = x^2(x^2 - 1) = x^2(x+1)(x-1)$
   b. Show that $p(x-1) - 12 = x^4 - 4x^3 + 5x^2 - 2x - 12$.
   SOLN: $p(x-1) - 12 = (x-1)^4 - (x-1)^2 - 12 = x^4 - 4x^3 + 6x^2 - 4x + 1 - (x^2 - 2x + 1) - 12$
   $= x^4 - 4x^3 + 5x^2 - 2x - 12$
   c. Complete the tables below (on separate paper):

x	-2	-1	$-1/2$	0	$1/2$	1	2
$p(x)$	12	0	$-3/16$	0	$-3/16$	0	12

x	-1	0	$1/2$	1	$3/2$	2	3
$p(x-1)-12$	0	-12	$-195/16$	-12	$-195/16$	-12	0

   Note that this is a shift 1 right and 12 down of *y* = *p*(*x*).
   d. Sketch graphs for $y = p(x)$ and $y = p(x-1) - 12$ together showing how one is a horizontal and vertical shift of the other.
   SOLN: (next page)



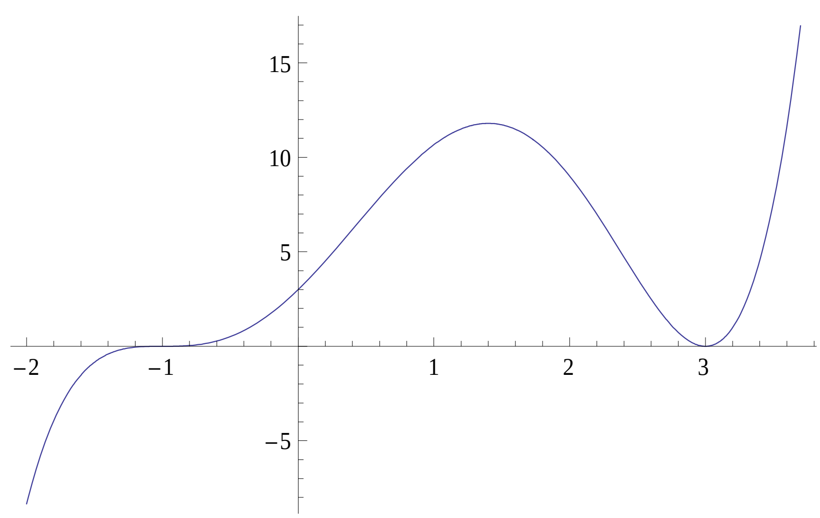
e. Describe the three roots for $y = p(x-1)-12$. How many are rational, how many are irrational and how many are not real? How do you know? Explain.
SOLN: From the graph and the table it is evident that $y = p(x-1)-12$ has rational zeros where $x = -1$ and $x = 3$. Thus $p(x-1)-12 = (x+1)(x-3)(x^2-2x+4)$ where x^2-2x+4 is an irreducible quadratic (discriminant = $(-2)^2 - 4(1)(4) = -12 < 0$. Thus the other two zeros are not real numbers.

3. Find a polynomial function with degree 3, integer coefficients and zeros at $x=-2,\ \ x=3+\sqrt{2}\,i$. Expand to descending powers form.
SOLN: By the conjugate zeros theorem, if *p* has real coefficients and a zero at $3+\sqrt{2}i$ then *p* must also have a zero at $3-\sqrt{2}i$. Thus, by the factor theorem, $p(x)=(x+2)\left[x-\left(3+\sqrt{2}i\right)\right]\left[x-\left(3-\sqrt{2}i\right)\right]$
We can use a difference of squares form to expand the product like so:

$$p(x)=(x+2)\left[(x-3)-\sqrt{2}i\right]\left[(x-3)+\sqrt{2}i\right]=(x+2)\left[(x-3)^2-\left(\sqrt{2}i\right)^2\right]=(x+2)\left[\left(x^2-6x+9\right)-2i^2\right]$$

$$=(x+2)\left(x^2-6x+11\right)=x^3-4x^2-x+22$$

4. Find a formula for the polynomial of degree 5 which has a root of multiplicity 3 at $x = -1$, a root of multiplicity 2 at $x = 3$ and a y-intercept at (0,3). Sketch a graph showing these key features.
SOLN: $p(x)=\frac{1}{3}(x+1)^3(x-3)^2$
satisfies all these requirements. The graph shown at right shows the behavior (flattening, but crossing where $x = -1$ (multiplicity 3) and just touching then turning around where $x = 3$ (multiplicity 2).



5. Consider $y = \dfrac{2x^2 + x - 1}{x^2 - 4}$.

a. Factor the numerator and denominator.

SOLN: $y = \dfrac{2x^2 + x - 1}{x^2 - 4} = \dfrac{(2x-1)(x+1)}{(x-2)(x+2)}$

b. What are the intercepts for this function?

SOLN: The intercepts are (0, ¼), (½,0) and (–1,0).

c. What are the vertical asymptotes?

SOLN: The lines $x = -2$ and $x = 2$ are vertical asymptotes.

d. What is the horizontal asymptote?

SOLN: The line $y = 2$ is a horizontal asymptote.

e. Plot additional points, as necessary, to get the shape of this function and sketch a graph.