Math 12 – Chapter 3 Test Solutions – Spring ’05

1. Consider the polynomial function $f(x) = 3x^4 + 2x^3 + 3x^2 - 4x - 4$.

   a. Describe how does the leading term determines the long-term behavior of this function; that is, what happens as $|x| \to \infty$.

   ANS: Since the leading coefficient is positive and the degree is even, as $|x| \to \infty$, $y \to \infty$.

   b. Evaluate $f(1)$ and use the result to find a factor of $f(x)$.

   ANS: $f(1) = 0$, so, by the factor theorem, we know $x - 1$ is a factor.

   c. Use synthetic division to divide $f(x)$ by $x - 1$ and relate dividend, divisor, quotient and remainder in an equation.

   ANS: The division algorithm finds the other factor to be such that

1	3	2	3	−4	−4
		3	5	8	4
	3	5	8	4	0

   $f(x) = (x-1)(3x^3 + 5x^2 + 8x + 4)$

   d. Compare the values of $f(-1)$ and $f(0)$ and use the intermediate value theorem to draw a conclusion about the location of a zero for $f(x)$.

   ANS: $f(-1) = 4$ and $f(0) = -4$, so, by the intermediate value theorem, there is zero between $x=0$ and $x=1$.

   e. State why the condition for the theorem on rational zeros is satisfied and use the theorem on rational zeros to list possible rational zeros for $f(x)$.

   ANS: Since the coefficients are all integers, the condition for the theorem on rational zeros is met. Therefore the rational zeros must be in $\pm\left\{1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right\}$.

   f. Use synthetic division to divide $f(x)$ by $x+2/3$ and relate dividend, divisor, quotient and remainder in an equation.

   ANS:

−2/3	3	2	3	−4	−4
		−2	0	−2	4
	3	0	3	−6	0

   So that $f(x) = \left(x - \frac{2}{3}\right)(3x^3 + 3x - 6) = (3x-2)(x^3 + x - 2)$.

   g. Use long division to divide $f(x)$ by $x^2 + x + 2$ and relate dividend, divisor, quotient and remainder in an equation.

   ANS:

$$
\begin{array}{r}
3x^2 - x - 2 \\
x^2 + x + 2 \overline{\smash{)}\, 3x^4 + 2x^3 + 3x^2 - 4x - 4} \\
-(\underline{3x^4 + 3x^3 + 6x^2}) \qquad\qquad \\
-x^3 - 3x^2 - 4x \qquad \\
-(\underline{-x^3 - x^2 - 2x}) \qquad \\
-2x^2 - 2x - 4 \\
-(\underline{-2x^2 - 2x - 4}) \\
0
\end{array}
$$

h. Find the complex conjugate zeros of x^2+x+2.

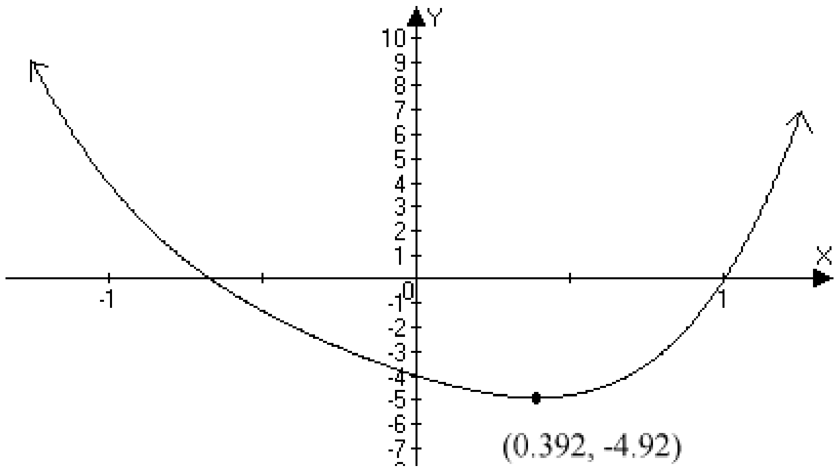
ANS: $x^2+x+2=\left(x+\frac{1}{2}\right)^2+\frac{7}{4}=0 \Leftrightarrow x=-\frac{1}{2}\pm\frac{\sqrt{7}i}{2}$

i. Write $f(x)=3x^4+2x^3+3x^2-4x-4$ in completely factored form.

ANS: $f(x)=(x-1)(3x+2)\left(x^2+2x+2\right)=(x-1)(3x+2)\left(x+\frac{1}{2}-\frac{\sqrt{7}i}{2}\right)\left(x+\frac{1}{2}+\frac{\sqrt{7}i}{2}\right)$

j. Sketch a graph of $f(x)=3x^4+2x^3+3x^2-4x-4$ showing how it passes through its intercepts and the coordinates of all local or global extrema.

ANS:



2. Explain why $p(x)=(x-5)\left(x^5+x^3+1\right)+1$ can have no zeros greater than 5.

ANS: If $x > 5$ then $x - 5 > 0$, $x^5+x^3+1>0$ so that $p(x) > 1$ and is therefore not 0.

3. Find a polynomial function with integer coefficients and zeros at $x=\frac{2}{3},\sqrt{2}i$.

ANS: Using the factor theorem and the conjugate roots theorem we have

$p(x)=(3x-2)\left(x-\sqrt{2}i\right)\left(x+\sqrt{2}i\right)=(3x-2)\left(x^2+2\right)=3x^3-2x^2+6x-4$

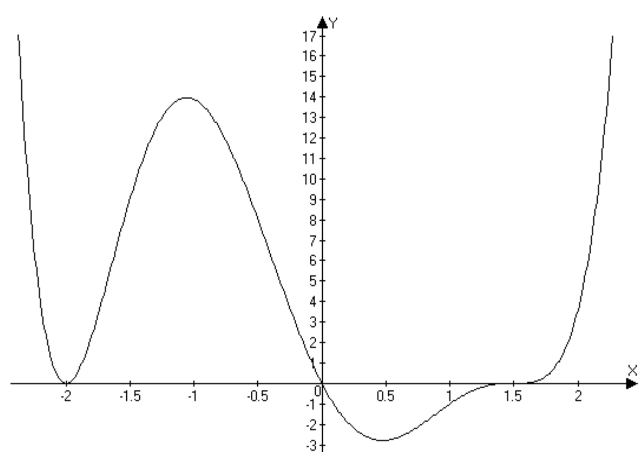
4. Find a formula for the 6th degree polynomial whose graph is shown. Hint: it passes through (1,-1) and the leading coefficient is less than 1.

ANS: There is a root of multiplicity 2 at $x = -2$, another root at $x = 0$ and a root of multiplicity 3 at $x = 1.5$, meaning that, by the factor theorem, $p(x)=ax(x+2)^2(2x-3)^3$.

To determine a, require that $p(1) = -1$, so that $-9a = -1$.

Thus $p(x)=\frac{1}{9}x(x+2)^2(2x-3)^3$

$=\frac{8}{9}x(x+2)^2\left(x-\frac{3}{2}\right)^3$

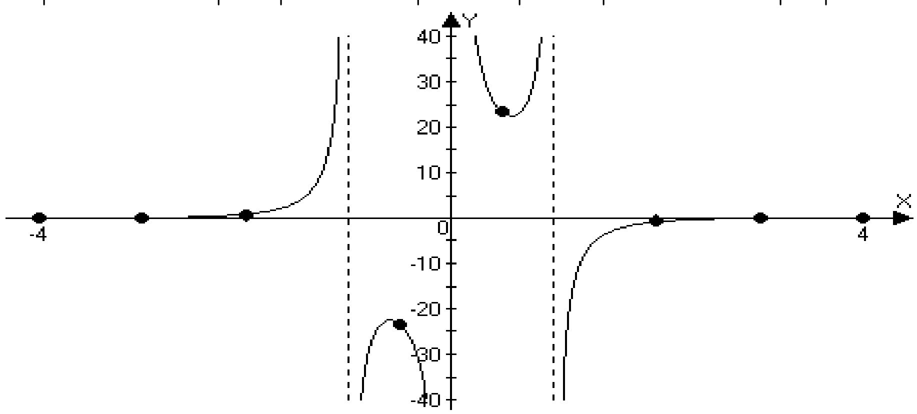


5. Sketch graphs for each of the following showing all intercepts, asymptotes and additional points, as necessary to get the shape.

   a. $y=\dfrac{x^2-9}{x\left(x^2-1\right)}$

   ANS: There are intercepts at (–3,0) and (3,0). The vertical asymptotes are along $x = -1$, $x = 0$ and $x = 1$. Since the degree of the numerator is less than the degree of the denominator, $y = 0$ is a horizontal asymptote. We plot a few additional points to flesh out the plot:

x	-4	-3	-2	$-1/2$	$1/2$	2	3	4
y	$\frac{-7}{60}\approx -0.12$	0	$\frac{5}{6}\approx 0.83$	$-23\frac{1}{3}$	$23\frac{1}{3}$	$-\frac{5}{6}\approx -0.83$	0	$\frac{7}{60}\approx 0.12$



   b. $y=\dfrac{\left(x^2-4\right)\left(2x^2-1\right)}{\left(x-1\right)^2\left(x^2+x+1\right)}$

   ANS: Intercepts at (–2,0),(2,0), $\left(\pm\frac{\sqrt{2}}{2},0\right)$ and (0,4). Vertical asymptote along $x = 1$ and a horizontal asymptote along $y = 2$. We tabulate additional points to flesh out the plot:

x	-8	-2	-1	$-.71$	0	0.5	$.71$	2	4	8
y	1.65	0	0.75	0	4	4.3	0	0	2.0	2.1