A Mathematician Catches a Baseball

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1. INTRODUCTION. In the game of baseball, what strategy does an outfielder employ to catch a fly ball? Recently, Michael McBeath and Dennis Shaffer, who are psychologists, and Mary Kaiser, a researcher at NASA, proposed a new model to explain how this task is accomplished [1]. The model, called the linear optical trajectory (LOT) model, was developed and tested empirically by the three researchers, and it received national attention during the 1995 baseball season [4]. In this paper, seeking to clarify what is written in [1] and [4], we develop equations relating the motion of a fly ball to the motion of an outfielder utilizing the LOT strategy. In the process, we provide a mathematical foundation on which the LOT model can rest.

To begin, let H be home plate, B the position of the ball, and F the position of the fielder at any point in time; see Figure 1. Define B^* to be the projection of B onto the playing field, so that H, F, and B^* are co-planar. There is a well-defined point I^*, which is the point of intersection of the line B^*F and the unique perpendicular to B^*F through H. There is another well-defined point I, which lies on the line BF, directly above I^*. The point I is the fielder's image of the ball.

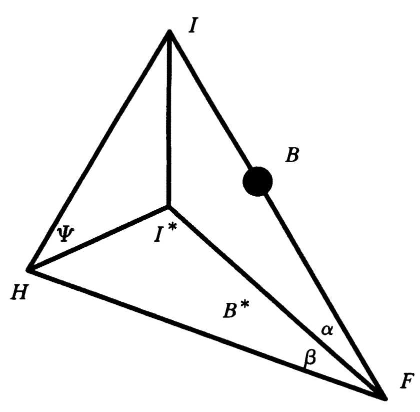

Figure 1

There are three important angles defined in the right pyramid $HFII^*$: the vertical optical angle $\alpha = \angle B^*FB$, the horizontal optical angle $\beta = \angle B^*FH$, and the optical trajectory projection angle $\Psi = \angle I^*HI$. We then have:

The LOT model hypothesis. *The strategy that a fielder uses to catch a fly ball is to follow a path that will keep the optical trajectory projection angle* Ψ *constant; this is equivalent to keeping the ratio* $(\tan \alpha)/(\tan \beta)$ *constant.*

In discussing the LOT model, McBeath and his colleagues write, "The LOT strategy discerns optical acceleration as optical curvature, a feature that observers are very good at discriminating," and, "If you're running along a path that doesn't allow the ball to curve down, then in a sense you are guaranteed to catch it." The LOT model apparently also applies to other situations, such as the pursuit of mates and prey by certain fish and houseflies. McBeath et al. also write, "[The LOT model] keeps the image of the ball continuously ascending in a straight line throughout the trajectory." [1,2] This last statement can be misleading to the casual reader who assumes it means that the trajectory of I must be linear. In Section 4, we clarify this statement and others that have been made about the model.

2. THE FIELDER, HIS PREY, AND THE IMAGE OF HIS PREY. If a fielder uses the LOT model, is his path uniquely determined by the path of the baseball? In this section, we develop equations relating the positions of the fielder, the ball, and the image of the ball, as we seek an answer to this question. Our analysis is in three-dimensional space, with the playing field represented by the xy-plane. Without loss of generality, let home plate H be the origin, and identify the first and third base lines with the x-axis and the y-axis, respectively, so that a fair ball is one that lands in the first quadrant of the plane. The coordinates of our three relevant points are $F = (x_f, y_f, z_f)$, $B = (x_b, y_b, z_b)$, and $I = (x_i, y_i, z_i)$. All nine coordinates are functions of time t (with $t = 0$ representing the moment that the ball is hit by the batter), and $z_f = 0$ at all points in time.

We define two other functions of time:

$$p = \frac{y_i}{x_i} \quad \text{and} \quad q = \frac{z_i}{x_i}. \tag{1}$$

If the trajectory of I is linear, then p and q would be constant. However, in the LOT model, this is not necessary. Instead, we have the following lemma:

Lemma 2.1. *If the LOT model is valid (i.e., if Ψ is constant), then $q^2/(1+p^2)$ is constant.*

Proof of Lemma 2.1: If we consider Figure 1, we see that

$$\tan^2 \Psi = \frac{|II^*|^2}{|HI^*|^2} = \frac{z_i^2}{x_i^2 + y_i^2} = \frac{q^2}{1+p^2}. \quad \blacksquare \tag{2}$$

It is helpful to first consider the case where p and q are constant, so we also introduce:

The strong LOT model hypothesis. *The strategy that the fielder uses to catch a fly ball is to follow a path that keeps both p and q constant.*

For either hypothesis, the line HI^* has slope p, so it follows that the line B^*F has slope $-1/p$; see Figure 2. Therefore, using the definition of slope, we have

$$-\frac{1}{p} = \frac{y_f - y_b}{x_f - x_b} \Rightarrow p = \frac{x_b - x_f}{y_f - y_b}, \tag{3}$$

and (3) is true at every point in time.

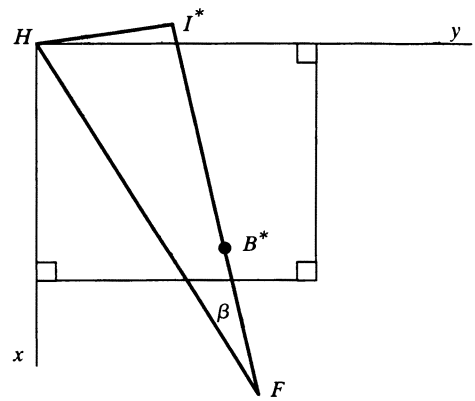

Figure 2

Next, we determine x_i in terms of B and p. The equation of the line HI^* is $y = px$, and the equation of the line B^*F is $y = y_b - (x - x_b)/p$. The point I^* is determined by the intersection of these two lines, and by setting these equations equal to each other and solving, we get

$$x_i = \frac{x_b + py_b}{p^2 + 1}. \tag{4}$$

Since F, B, and I are collinear, we have $(z_i - z_f)/(x_i - x_f) = (z_b - z_f)/(x_b - x_f)$, and since $z_f = 0$ and $z_i = qx_i$, solving this equation for x_f gives

$$x_f = x_i\left(\frac{z_b - qx_b}{z_b - qx_i}\right). \tag{5}$$

Combining (4) and (5) and simplifying yields

$$x_f = \frac{(z_b - qx_b)(x_b + py_b)}{z_b(p^2 + 1) - q(x_b + py_b)}. \tag{6}$$

Then, combining (3) and (6) and simplifying leads to

$$y_f = \frac{(pz_b - qy_b)(x_b + py_b)}{z_b(p^2 + 1) - q(x_b + py_b)}. \tag{7}$$

Finally, if we solve (6) for q, we get

$$q = \left(\frac{z_b}{x_b + py_b}\right)\left(\frac{x_f(p^2 + 1) - (x_b + py_b)}{x_f - x_b}\right). \tag{8}$$

What we now have, for every $t > 0$ and for every trajectory B, is a relationship between (x_f, y_f) and (p, q). If we know p and q, then (6) and (7) yields x_f and y_f, and if we know x_f and y_f, then (3) and (8) give us p and q. In the next two sections, we use these relationships to investigate the subtle elements of the LOT model.

3. THE STRONG LOT MODEL HYPOTHESIS: CONSEQUENCES AND PROBLEMS. For a given ball trajectory B, let T be the time when the ball is either caught or hits the ground, so that the ball is in flight for $0 < t < T$ and $z_b = 0$ when $t = T$. We assume that $x_b + py_b \nrightarrow 0$ as $t \to T$.

Theorem 3.1. *For a given ball trajectory B, and for every t_0 such that $0 < t_0 < T$, there exists a unique fielder's path, depending on the position of the fielder at time t_0, such that the fielder can use the strong LOT model for $t_0 \le t \le T$ on that path and catch the ball at time T.*

Proof of Theorem 3.1: At time t_0, the positions of the fielder and the ball are known. Therefore, using (3) and (8), we can determine p and q. Once these two constants are known, (6) and (7) specifies the unique path the fielder follows. At time T, we have $z_b = 0$, and therefore

$$x_f|_{t=T} = \left.\frac{(z_b - qx_b)(x_b + py_b)}{z_b(p^2+1) - q(x_b + py_b)}\right|_{t=T} = \left.\frac{-qx_b(x_b + py_b)}{-q(x_b + py_b)}\right|_{t=T} = x_b|_{t=T}. \quad (9)$$

Similarly, $y_f|_{t=T} = y_b|_{t=t}$. ■

Thus, the strong LOT model works, provided, of course, that the fielder can run fast enough to follow his predetermined path. Can a fielder track a ball starting at the moment the ball is launched by the batter? McBeath and his colleagues write, "...fielders do not and cannot arbitrarily select optical angles and rates of change... but rather they maintain the initial optical projection angle, Ψ, which is fully determined by the perspective launch angle of the ball relative to the fielder." [2] This suggests that formulas can be developed for p and q, and hence Ψ, from the information at $t = 0$, and that perhaps the strong LOT strategy can be utilized from the moment the ball is hit.

Lemma 3.1. *Under the assumption that p and q are constant near $t = 0$, the values of p and q are uniquely determined at $t = 0$ by the initial velocity of the ball and the initial position of the fielder.*

Proof of Lemma 3.1: Since (3) is true for all t, it is true for when the batter hits the baseball ($t = 0$), therefore we can conclude that

$$p|_{t=0} = \left.\frac{x_b - x_f}{y_f - y_b}\right|_{t=0} = -\left.\frac{x_f}{y_f}\right|_{t=0}. \quad (10)$$

To determine q, we use (8) and L'Hôpital's rule:

$$q|_{t=0} = \lim_{t\to 0}\left(\frac{z_b}{x_b + py_b}\right)\left(\frac{x_f(p^2+1) - (x_b + py_b)}{x_f - x_b}\right)$$

$$= \lim_{t\to 0}\left(\frac{z_b}{x_b + py_b}\right)(p^2+1) = \left.\frac{z_b'(p^2+1)}{x_b' + py_b'}\right|_{t=0}. \quad (11)$$

Thus, (10) and (11) provide formulas for p and q as functions of the position of the fielder and the velocity vector of the ball at $t = 0$. ■

Lemma 3.1 clearly applies to the strong LOT model, but, as we see in the following theorem, we now have a problem with the strong LOT strategy at $t = 0$, because q does not depend on the initial position of the fielder.

Theorem 3.2. *For a given ball trajectory B, there exist points $(\bar{x}_f, \bar{y}_f, 0)$, called* ideal fielders' positions, *such that a fielder situated at that position when the ball is hit can* *use the strong LOT model for $0 \leq t \leq T$ to determine a unique path to catch the ball. Not all fielders' positions are ideal fielders' positions.*

Proof of Theorem 3.2: Choose an arbitrary value of p. From Lemma 3.1, we can determine q as a function of B and p and define

$$\bar{x}_f = \lim_{t \to 0} \frac{(z_b - qx_b)(x_b - py_b)}{z_b(p^2 + 1) - q(x_b + py_b)} \tag{12}$$

and

$$\bar{y}_f = \lim_{t \to 0} \frac{(pz_b - qy_b)(x_b + py_b)}{z_b(p^2 + 1) - q(x_b + py_b)}. \tag{13}$$

A fielder situated at $(\bar{x}_f, \bar{y}_f, 0)$ at $t = 0$ can then use the unique path described by (6) and (7) in order to catch the ball at time T.

Now fix $\lambda \neq 1$ and consider a second fielder situated at $(\lambda\bar{x}_f, \lambda\bar{y}_f, 0)$ when $t = 0$. This fielder has the same p and q as the first fielder (due to Lemma 3.1), and therefore the same running path as the first fielder, which is impossible. The second fielder is not in an ideal fielder's position. ■

As a practical matter, since q depends on the velocity vector of the ball, a fielder would need some period of time after the ball was hit to recognize what Ψ is. This means an ideal fielder is ideal in another sense, because at $t = 0$, he can instantaneously discern the velocity of the baseball and begin his path to catch the ball. Theorem 3.1, which doesn't apply when $t = 0$, is a better description of how the strong LOT model operates, and it is simply incorrect to say that a fielder can use the LOT strategy from the crack of the bat. It makes more sense to say that fielders maintain a constant Ψ, which is determined a moment after the ball is launched by the batter.

4. ONE PATH OR MANY: USING MATHEMATICS TO CLARIFY IDEAS. In the case where we use the LOT model and not the strong LOT model, there is no longer a unique path that a fielder must follow in order to achieve a linear optical trajectory.

Theorem 4.1. *For a given ball trajectory B, and for every t_0 such that $0 < t_0 < T$, there exist an infinite number of fielders' paths, such that a fielder can use the LOT strategy for $t_0 \leq t \leq T$ on that path and catch the ball at time T.*

Proof of Theorem 4.1: This theorem is proved the same way as Theorem 3.1, except now, as t increases, both p and q are allowed to vary, as long as, by Lemma 2.1, $q^2/(1 + p^2)$ remains constant. This gives an infinite number of solutions. ■

With the assumption of Lemma 3.1, we can also define ideal fielders' positions for the LOT model. The proof is the same as Theorem 4.2, except that now the path determined by (12) and (13) is not unique.

We can use our results to examine the validity of, and explain, several statements from [1] and [2]. The benefit of the mathematical analysis is that we can recognize how these statements follow from the LOT model and gain a greater appreciation for the model. Here are the statements:

A. ...the [LOT model] strategy in itself does not specify a unique solution. [2]
B. ...angle of bearing appears to be used as an additional constraint to help determine the particular LOT chosen. [2]
C. "[The LOT model] keeps the image of the ball continuously ascending in a straight line throughout the trajectory." [1]
D. One interesting aspect that has emerged from research on this problem is that for identical launches, fielders will select different running paths, particularly near the beginning and end of the task. A good model of outfielder behavior should allow for this variability, as the LOT strategy does. Near the beginning of the trajectory [of the ball], we expect more variability because outfielder location has less influence on the optical trajectory. Near the end we expect more variability because corrective action will commence as other depth cues become available. [2]

Statement A is a result of Theorem 4.1. The angle of bearing in statement B is a function of p and q, and since q is a function of p by Lemma 2.1, it makes sense to call p the *fielder's bearing function.* As the fielder tracks the ball, he unconsciously chooses a function p as a part of his LOT strategy. (The strong LOT model keeps the angle of bearing constant.) Since it is reasonable to assume that a player's bearings will not change in the first moment after the ball is hit, we are justified in keeping p constant near $t = 0$ in Lemma 3.1.

Statement C seems at odds with the idea that p and q can be allowed to vary, which suggests that the LOT hypothesis does not allow the trajectory of I to be non-linear. A fielder's bearings *can* change, and this corresponds to a rotation about home plate of the right pyramid in Figure 1. For example, if the bearing function changes from a value of p_1 to a value of p_2, then the angle of rotation is $\theta = \arctan p_2 - \arctan p_1$. From the rotation of axes formulas, we get

$$\cos\theta = \frac{1 + p_1 p_2}{\sqrt{(1+p_1^2)}\sqrt{(1+p_2^2)}} \quad \text{and} \quad \sin\theta = \frac{p_2 - p_1}{\sqrt{(1+p_1^2)}\sqrt{(1+p_2^2)}}. \tag{15}$$

Suppose we have a situation where first we have $I_1 = (x_i, p_1 x_i, q_1 x_i)$ and then a change of bearings leading to $I_2 = (x_i, p_2 x_i, q_2 x_i)$. Applying the change of variables formula to I_2, we end up with the rotated point

$$I_2' = \sqrt{\frac{1+p_2^2}{1+p_1^2}}\,(x_i, p_1 x_i, q_1 x_i). \tag{16}$$

From the perspective of the fielder, the ball appears to remain on the vector $\langle 1, p_1, q_1 \rangle$, "continuously ascending on a straight line."

Statement D is a consequence of the proof of Theorem 4.1, which explains how there can be many paths that keep Ψ constant. Also, from (3), we get

$$\frac{\partial p}{\partial x_f} = -\frac{1}{y_f - y_b} \quad \text{and} \quad \frac{\partial p}{\partial y_f} = -\frac{x_b - x_f}{(y_f - y_b)^2}. \tag{17}$$

These derivatives are small when t is near 0, as are the derivatives for q. As a consequence, when (3) and (8) are used to determine p and q "near the beginning of the trajectory," fielders near each other have quite similar optical trajectory projection angles Ψ, hence "outfielder location has less influence on optical trajectory." Therefore, different running paths can keep p and q nearly constant, as the LOT model predicts.

The mathematics presented here shows that the LOT model is reasonable and, interestingly, qualitative observations made by the researchers can be supported quantitatively by the analysis. This analysis, though, *does not prove* that the LOT model is correct. The LOT model was developed by perceptual psychologists using statistical methods and it cannot be proved as we prove a theorem in mathematics. It also goes without saying that outfielders do not and cannot follow the LOT strategy *exactly*, or even that outfielders follow the strategy well. In [1], McBeath et al. reported that one fielder appeared to use a linear optical trajectory for a while, faltered, then continued on a new optical trajectory with a different Ψ!

There are other, competing models, such as the optical acceleration cancellation (OAC) model, that have their defenders. According to the OAC model, a fielder acts to keep $d(\tan\alpha)/dt$ constant, not Ψ. Another view comes from Robert K. Adair, a physicist, who argues that "a fielder runs laterally so that the ball goes straight up and down from his or her view." [3] The OAC model or Adair's may be correct, although McBeath and his colleagues rebutted both theories with this statement, "Both maintenance of lateral alignment and monitoring of up and down ball motion require information that is not perceptually available from the fielder's vantage." [2]

An example of the failure of the LOT strategy was recently presented by James L. Dannemiller, Timothy G. Babler, and Brian L. Babler. They write that it is possible for a fielder to use the LOT strategy and arrive "away from the ball's landing site at the instant the ball hits the ground." [5] This *is* possible if the fielder chooses a path such that $x_b + py_b \to 0$ as $t \to T$. In this case, Theorems 3.1 and 4.1 are invalid and $B \neq F$ when $t = T$. However, in this instance, H, B, and F would become collinear at the moment the ball hits the ground.

5. A MATHEMATICIAN CATCHES A BASEBALL. We now go to the ball park, and a mathematician on the visiting team is standing in right field, waiting to catch some fly balls for his team. It is the bottom of the ninth inning, and the visitors are ahead 4–3. The ball is hit! Let's suppose that B^*, the projection of the ball on the field, moves in a straight line and that the path of the ball is a parabolic arc. These are reasonable assumptions (unless we are playing in Wrigley Field, where it is rather windy at times) and an example of a ball's trajectory (in feet) would be

$$x_b(t) = 75t, \qquad y_b(t) = 10t, \qquad z_b(t) = -64t^2 + 256t. \tag{18}$$

This is a ball hit to deep right field that will be in the air for 4 seconds and will land approximately 303 feet from home plate unless our intrepid mathematician gets there in time.

Suppose that our right fielder is positioned in right-center field at the crack of the bat at the position $(x_f, y_f, z_f) = (270, 70, 0)$, that it takes him 0.3 seconds to get his bearings, and that he plans to use the strong LOT model to catch the ball. Utilizing Theorem 3.1, he determines that

$$p \approx -3.694 \qquad \text{and} \qquad q \approx 99.121. \tag{19}$$

Therefore, $\Psi \approx 87.8°$; this is practically vertical—perhaps Adair has a point! Now that p and q are known, a unique path can be determined for our all-star to catch this fly ball. That path is indicated in Figure 3.

On the very next play, inexplicably another ball is hit to right field, with the same trajectory and the same response time for our athlete. This time, though, he decides to use the regular LOT strategy, computing his initial p and q as above, and then using $p = -0.0827t - 3.6692$ as his bearing function. This path, which

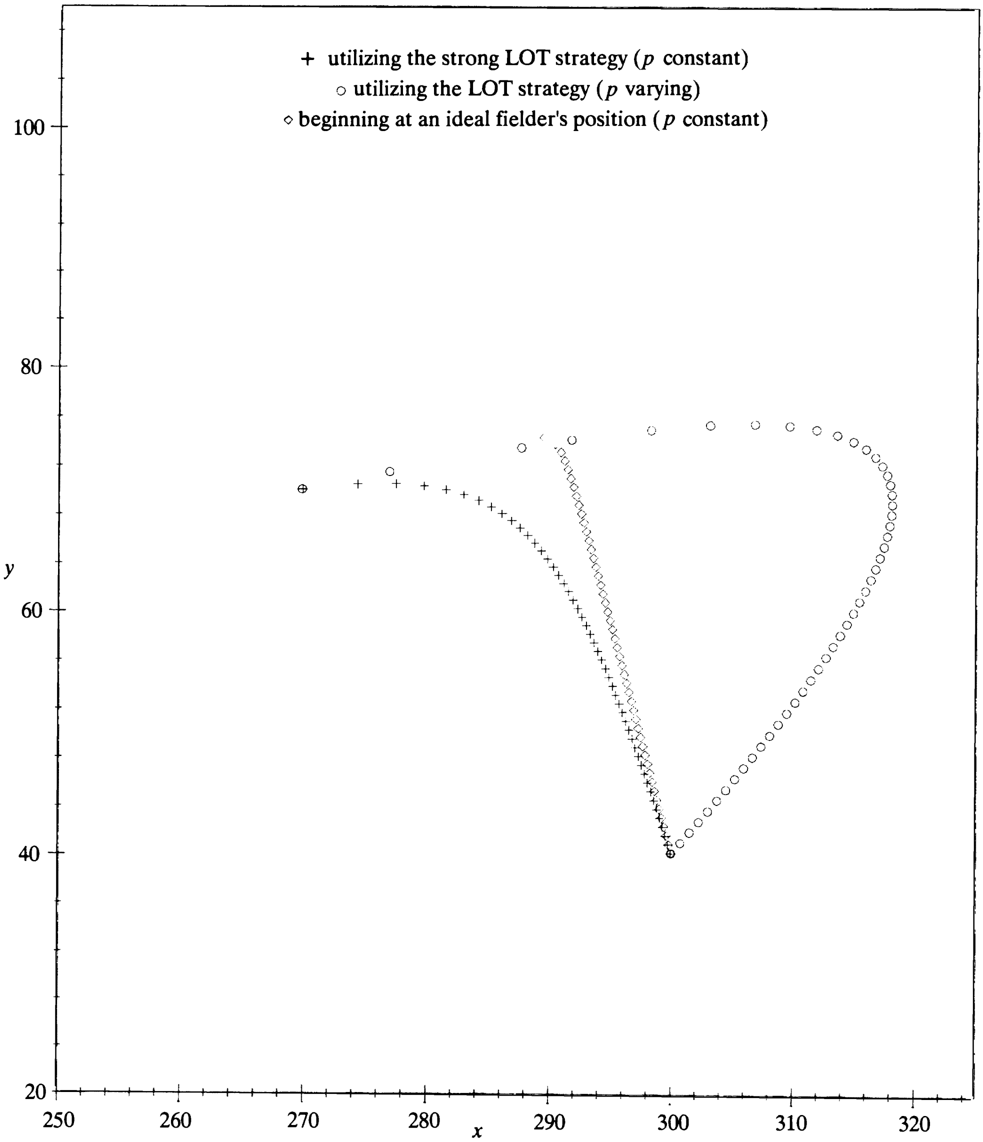

Figure 3

sends the fielder to the edge of the playing field and back, is also indicated in Figure 3.

Now that there are two outs, our mathematician wonders if the next batter will also hit a fly ball satisfying (18). He decides to get into an ideal fielder's position. Although the most obvious one is (300, 40, 0)—the point where the ball has landed the first two times—he decides to find an ideal position that corresponds to his current position. He computes $p \approx -3.857$, using (10), and $q \approx 111.579$, using (8). Then, making use of equations (12) and (13), and a little calculus, he determines

$$\bar{x}_f \approx 291 \quad \text{and} \quad \bar{y}_f \approx 75. \tag{20}$$

Figure 4 shows, for a ball following trajectory (18), several ideal fielder's positions, depending on the choice of p. Again in Figure 3, we see out hero,

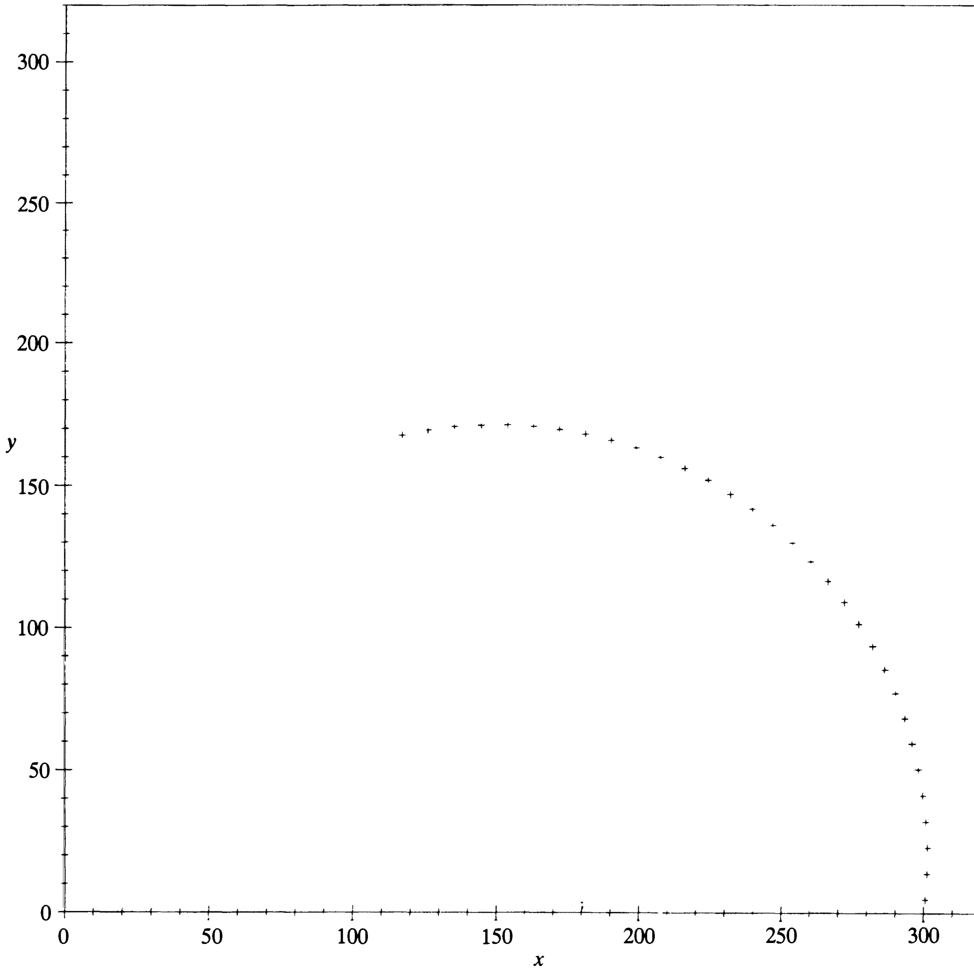

Figure 4

starting at (20), pursuing the ball using the strong LOT strategy and catching it after 4 seconds. That's three outs—game over!

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