CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

A Geometric Series from Tennis

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During the Wimbledon tennis finals, the commentator mentioned that one of the players was winning 60% of his points on serve. I began wondering what fraction of the games a person should win if the probability of winning any particular point was p. In answering this question, I used some basic probability and summed a geometric series. Others might want to share this with their students.

As a reminder, the winner of a game is the first person to score 4 points, unless the game reaches a 3-to-3 tie. Then it continues until someone goes ahead by 2. (The four points are called 15, 30, 40 and game, by the way.)

Assume that the probability that player A wins a point is p. The probability that player A wins the game within the first 6 points is the probability that A leads by a score of 3-to-0, 3-to-1, or 3-to-2, and then wins the next point, which is

$$\left[\binom{3}{0}p^3 + \binom{4}{1}p^3(1-p) + \binom{5}{2}p^3(1-p)^2\right]p = p^4(15 - 24p + 10p^2),$$

after simplification. If neither player has won by the 6th point, then the score must be tied at 3-to-3. The probability of a 3-to-3 tie is

$$\binom{6}{3}p^3(1-p)^3 = 20p^3(1-p)^3.$$

After that, the game must be won after an even number of points. The probability of winning on the 8th point is the probability of a 3-to-3 tie, followed by winning the next two points, which is

$$20p^3(1-p)^3p^2.$$

The probability of winning on the 10th point is the probability of a 3-to-3 tie, splitting the next two points, and then winning the 9th and 10th points, which is

$$20p^3(1-p)^3[2p(1-p)]p^2.$$

More generally, the probability of winning on the $(2n+6)$th point is the probability of a 3-to-3 tie, $20p^3(1-p)^3$, splitting each pair of the next $2n-2$ points $[2p(1-p)]^{n-1}$, and then winning the last 2 points p^2, which is

$$20p^3(1-p)^3[2p(1-p)]^{n-1}p^2.$$

Thus, the probability of A's winning the game is

$$f(p) = p^4(15 - 24p + 10p^2) + 20p^3(1-p)^3 p^2 \sum_{n=0}^{\infty} [2p(1-p)]^n.$$

Summing the geometric series gives the function

$$f(p) = p^4(15 - 24p + 10p^2) + \frac{20p^5(1-p)^3}{1-2p(1-p)},$$

whose graph is shown in Figure 1.

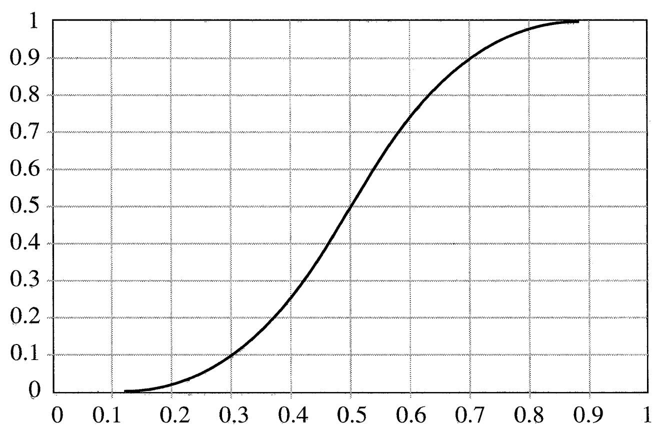

Figure 1. Graph of probability function f of winning a game as function of the probability p of winning a point.

The answer to my original question is that players who win 60% of their points on serve will win $f(0.6) \approx 0.74$ or about 74% of their service games.

The graph suggests that f has rotational symmetry about the point $(0.5, 0.5)$, although this is not apparent from the form of f. We can verify the rotational symmetry by noting that if q is the probability that the second player wins a point, then $f(q)$ gives the probability that the second player wins the game. But $p + q = 1$, so $f(q) = f(1-p)$. Since one of the two players must win, $f(p) + f(1-p) = 1$. If we take $p = 0.5 + x$ and $g(x) = f(0.5 + x) - 0.5$, we find that

$$g(x) + g(-x) = f(0.5 + x) + f(0.5 - x) - 1 = f(p) + f(1-p) - 1 = 0,$$

so $g(-x) = -g(x)$. Thus, g is an odd function, and hence f has rotational symmetry.

From the graph, it also appears that there is a point of inflection at $p = 0.5$. It would be tedious to check that $f''(0.5) = 0$. However, if we remember the derivation

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

and apply that to $g(x)$, we see that

$$f''(0.5) = g''(0) = \lim_{h\to 0} \frac{g(h) + g(-h) - g(0)}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0,$$

which shows that $p = 0.5$ gives a point of inflection. This means that players gain the most in the number of games they win by increasing p when $p \approx 0.5$. In other words, if you are a weak or a strong player relative to your opponent (p small or large), then a small improvement in your serve (increasing p) doesn't result in as much improvement in the number of games you win as an improvement against a comparable opponent (p about 0.5) does. A simple computation shows that $f'(0.5) = 2.5$, so a 1% gain in p will result in about a 2.5% increase in your likelihood of winning a game.

On Sums of Cubes

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A well-known identity for the sum of the first n cubes is

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2. \tag{1}$$

Some of our students noticed that, curiously, equality still holds if $n - 1$ is replaced by 2, that is

$$1^3 + 2^3 + \cdots + (n-2)^3 + 2^3 + n^3 = (1 + 2 + \cdots + (n-2) + 2 + n)^2.$$

This observation led us to ask whether other such switches are possible. In this note, we investigate those triples (k, m, n) for which

$$\sum_{j=1}^{k-1} j^3 + m^3 + \sum_{j=k+1}^{n} j^3 = \left(\sum_{j=1}^{k-1} j + m + \sum_{j=k+1}^{n} j\right)^2. \tag{2}$$

Clearly (2) holds (because of (1)) if $m = k$, so in what follows we assume $m \neq k$. Furthermore, as our students pointed out, $(n - 1, 2, n)$ is a solution for every $n \geq 2$. Do other solutions exist, and if so, do they fit nice patterns? Our inquiry led us to some interesting and unexpected answers, and ultimately to a connection with an unsolved problem in number theory.

Our first result gives a necessary and sufficient condition for a triple (k, m, n) to be a solution of (2). (We assume that k and m are positive integers not exceeding n.)

Theorem 1. *A triple (k, m, n) with $m \neq k$ satisfies (2) if and only if either*

(a) $k = n - 1$ *and* $m = 2$, *or*

(b) *there exists an integer $p \geq 2$ and a positive divisor s of $3p(p-1)$ for which*

$$(k, m, n) = \left(2(p-1) + \frac{3(p-1)p}{s}, 2p + s, m + k - p\right). \tag{3}$$

Furthermore, different pairs (p, s) yield different solutions.