Designs, Geometry, and a Golfer's Dilemma

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I was taken off guard the other day when my father-in-law, John, posed to me a very simply stated problem. He plays golf. In fact, John plays a lot of golf. When you play as much golf as he does, you become bored playing with the same people over and over again. So here's the problem: John regularly plays with a group of 16 people. Three days a week for the entire summer, they go out in 4 groups of 4 players each to hit the course. Is there some way they can arrange the players in the groups each day so that everybody plays with everybody else in some sort of regular way? As my father-in-law said, "We want to mix it up as much as possible."

First of all, we need to figure out what the question is asking. Let's look at the problem from the perspective of my father-in-law. Suppose that John plays his first day with three other players, say Keith, Bill, and Howard. Then, there are still 12 other people available to play. John would prefer to play with all 12 other people before he ends up playing with Keith, Bill, or Howard again. From John's perspective, this seems like it may not be a difficult problem. Simply assign three players to John for the first day, three different players to John for the second day, etc. Then, after 5 days, John will have played with all of the other 15 people in the group. However, remember that we need to assign 4 groups (not just John's group) of 4 players each, and we want *every* golfer to play with *every other* golfer, again in some sort of regular way. Suddenly the problem seems much more difficult. In the next several pages, we will find some solutions to the problem. In our quest to find a best solution we will take a ride through some areas of discrete mathematics including finite affine and projective planes, and combinatorial designs.

A connection to affine planes

One basic solution to the golfer's dilemma comes from a rather unexpected area of mathematics, geometry. How could this be relevant? Bear with me for a few paragraphs and we'll get back to golfing soon enough. We need some terminology. The formal definition of an affine plane goes like this.

DEFINITION. An **affine plane** is a set of points together with a collection of subsets of these points, called lines, such that

1. every two distinct points determine a unique line,
2. if l is a line and P is a point not on l, then there exists a unique line m such that P is on m and l and m have no points in common, and
3. there exist 3 noncollinear points.

(You can read about playing tic-tac-toe on affine planes in Carroll and Dougherty's article in this issue of the MAGAZINE.) One important point, which is made clear by the second axiom, is that affine planes have *parallel lines*. This may not seem like a big deal, but in the world of higher mathematics, people do without parallel lines all the time. In fact, we will soon see another kind of plane where parallel lines do not exist. Now we add an additional condition. Suppose that we have an affine plane $\mathcal{A}$ that contains only a finite number of points. Is this possible? Indeed it is.

Consider a 2-dimensional vector space V over some field $\mathcal{F}$. We define points to be all of the vectors of V and define lines to be all of the cosets of all of the 1-dimensional subspaces contained in V. For example, take V to be the vector space $\mathbb{R}^2$ whose vectors are all ordered pairs (x, y) for $x, y \in \mathbb{R}$. The cosets of the 1-dimensional subspaces are sets of the form $\{\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\}$ for some $\mathbf{u}, \mathbf{v}$ in V. These cosets of V are exactly what we typically call the lines of the coordinate plane. Hence, a 2-dimensional vector space can be used to model an affine plane.

Now, we again consider V as a 2-dimensional vector space, but this time restrict the coordinates of the vectors of V to be in the finite field $GF(q)$ that contains q elements. The notation $GF(q)$ means the *Galois field* with q elements, named after the French mathematician Evariste Galois (1811–1832). For those unfamiliar with finite fields, simply think of the coordinates x and y as coming from a finite set that only contains q elements. One can prove that the number of elements in a finite field is always a power of some prime number. Hence, we refer to q as being a *prime power*. Again, considering all of the vectors of V as points and all of the cosets of all of the 1-dimensional subspaces as lines, we obtain an affine plane. This time, however, our affine plane contains only a finite number of points, namely, the number of vectors of V. Since each vector is written as an ordered pair of elements from $GF(q)$, we see that V contains q^2 vectors. By varying t in the definition of cosets given above, we see that the number of points on a line is equal to the number of elements in the finite field. Hence, every line contains exactly q points.

We can do some more involved counting to find other properties of our affine plane. For instance, fix a vector $\mathbf{v} \in V$ and count how many cosets of 1-dimensional subspaces pass through $\mathbf{v}$. To do this, we note that there are $q^2 - 1$ choices for a second vector $\mathbf{w}$ different from $\mathbf{v}$. The vectors $\mathbf{v}$ and $\mathbf{w}$ together determine a coset of a 1-dimensional vector subspace, say C. But this coset C could be determined from $\mathbf{v}$ and *any* other vector in C. Since there are $q - 1$ choices for another vector in C, each such coset has been counted $q - 1$ times. Therefore, the total number of cosets of 1-dimensional subspaces through the given vector $\mathbf{v}$ is exactly $(q^2 - 1)/(q - 1) = q + 1$. Hence, every point lies on exactly $q + 1$ lines.

Finally, the number of lines can be counted by counting the number of ways to choose two distinct points to generate a line, and then dividing by the number of ways any given line was counted. The number of ways to choose an ordered pair of two distinct vectors is $q^2(q^2 - 1)$. But each line is generated by choosing *any* such pair of points on that line, which can be done in $q(q - 1)$ ways. Hence, the total number of lines is exactly $(q^2(q^2 - 1))/(q(q - 1)) = q^2 + q$. The affine plane obtained from this model of a 2-dimensional vector space over the finite field $GF(q)$ is denoted $AG(2, q)$ (the classical affine geometry of dimension 2 and order q).

We can say a little more. Note that two cosets of the same 1-dimensional subspace of a vector space never intersect. Hence, we have collections of lines in our affine plane no two of which meet. Such sets of lines are naturally called *parallel classes of lines*. Counting can again be used to show that each parallel class contains exactly q lines. Since there are a total of $q(q + 1)$ lines, there must be $q + 1$ different parallel classes.

Let's get back to the original problem of the golfer's dilemma. We have a total of 16 golfers that we want to break into various groups of 4. Now, let $q = 4$ in the affine plane model above. The affine plane $AG(2, 4)$ contains exactly 16 points, and every line contains exactly 4 points. Every parallel class contains exactly 4 lines, and there are exactly 5 parallel classes. Hence, we have a solution to the golfer's dilemma by letting the points of $AG(2, 4)$ represent the golfers, and the lines represent the various groups of 4 golfers playing together. The parallel classes of lines represent the various days of play since each parallel class consists of 4 distinct groups of 4 players each (that is, 4 parallel lines in each parallel class).

When the finite field is relatively small, one can try to find the lines of $AG(2, q)$ by hand. A software package such as *Magma* [**3**] does this computation virtually instantaneously, but since $q = 4$ is pretty small, let's get started doing it by hand. Keep in mind that since this is a *finite* affine plane, lines can be thought of simply as subsets of points with little relation to shapes. First we write the 16 players in a 4×4 grid as in FIGURE 1.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 1 Representing $AG(2, 4)$

The rows and columns of the grid can each represent a parallel class. This means that any further lines must contain exactly one point from each row, and one point from each column (since two points in the same row or column would uniquely determine one of the lines already given). At this point we might try using the diagonals to get two more lines. The reader is encouraged to try to find the remaining lines by hand before proceeding, the lesson being that *this is not at all easy*. The use of finite geometry (along with a little computer power to generate the cosets of the appropriate vector subspace) gives us the blocks in TABLE 1.

TABLE 1: A five-day schedule for 16 golfers.

Day 1 (rows)	Day 2 (columns)	Day 3 (diagonals)	Day 4	Day 5
{1, 2, 3, 4}	{1, 5, 9, 13}	{1, 6, 11, 16}	{1, 7, 12, 14}	{1, 8, 10, 15}
{5, 6, 7, 8}	{2, 6, 10, 14}	{2, 5, 12, 15}	{2, 8, 11, 13}	{2, 7, 9, 16}
{9, 10, 11, 12}	{3, 7, 11, 15}	{3, 8, 9, 14}	{3, 5, 10, 16}	{3, 6, 12, 13}
{13, 14, 15, 16}	{4, 8, 12, 16}	{4, 7, 10, 13}	{4, 6, 9, 15}	{4, 5, 11, 14}

We have solved the problem of the golfer's dilemma: Assign each golfer a number between 1 and 16. Then, over the course of 5 days, the golfers play together based on the schedule outlined in TABLE 1. At the end of 5 days, every golfer will have played with every other golfer exactly once.

The golfers aren't happy

Having found a solution to the problem, I was quick to email a solution to my father-in-law, but was rather disappointed at his immediate response. First of all, we have only covered 5 days of play. These guys want to play *all summer*. So what do we do? A natural remedy is to simply repeat the process. That is, after 5 days, just start over with day 1. That way, after, say, 25 days of play, every golfer will have played with every other golfer exactly 5 times. However, there is a clear disadvantage to repeating our solution.

Let's go back to the beginning. Suppose that John plays his first day with Keith, Bill, and Howard. Then, 5 days later, the golfers all decide to repeat the schedule. When

John plays with Keith for the second time, the other two members of their group will again be Bill and Howard. It would be nice if John and Keith could play together with two *different* people the next time around. More precisely, we see that two distinct golfers uniquely determine a group. That is, if I pick any two golfers, say John and Keith, from the group of 16, there is exactly one group of 4 in which John and Keith are both members. In our example, it is the group that contains Bill and Howard. From the perspective of the affine plane, this really comes as no surprise. Recall that two points of the affine plane determine *exactly one* line.

So let's kick it up a notch. Here's one quick and easy way to remedy the situation. Assign each golfer a number between 1 and 16, and play through the five day schedule as outlined in TABLE 1. After the five days are up, permute the numbers in some way, and then repeat the schedule. The golfers could all pick a partner to switch numbers with, or they could cyclically shift their numbers ($i \to i+1$ for i between 1 and 15, and $16 \to 1$). Of course, one must be careful with such a cyclic shift. The reader should check that after certain cyclic shifts, the groups will start to repeat. Is there some more systematic way to ensure that every golfer plays with every other golfer, but eliminate the drawbacks of the solution already given?

Statisticians face these sorts of questions all the time when they are designing experiments. They have a set of v objects on which they want to run an experiment, but the experiment can only be run on k objects at a time. In our case, $v = 16$ and $k = 4$, and maybe our experiment consists of determining the ability of each golfer. The statisticians want to mix things up as much as possible. Maybe object 2 could affect the outcome of the experiment on object 1; maybe Bill makes John nervous. So, in order to get an accurate reading on John's golf ability, we need to make sure that Bill doesn't play with John every single time. In a similar fashion, suppose Bill alone doesn't make John nervous, but when Bill and Howard get together, they goof around a lot and it makes John nervous. So, it would be OK to put these three together once, but if John and Bill are together again, it would be best if Howard isn't included the second time around. More generally, *we would like every set of three golfers to be grouped together exactly once*. Can this be done?

First we note that if every two golfers are together exactly once, then the schedule would run in 5 days (as discussed above). This makes sense simply by counting. That is, if John plays with 3 different people each day, it would take 5 days for him to play with all of the remaining 15 players. Can we apply the same reasoning to the new problem? That is, suppose every group of 3 golfers play together exactly once. How long will the schedule last? From John's perspective, the answer is equal to the number of groups in which John is a member. But remember, *three* golfers determine a group now. The number of ways to choose 2 golfers from the remaining 15 is $\binom{15}{2} = 105$. Once two other golfers are chosen, John and the two others uniquely determine the group, say, John, Keith, Bill, and Howard. But whether we pick Keith and Bill, Keith and Howard, or Howard and Bill as the additional two golfers, we will always get the same group. Hence, each such group is counted 3 times. Therefore, the number of groups in which John is a member is $105/3 = 35$. So the schedule would last for 35 days, or about 12 weeks if they play 3 days per week. This would cover most of the summer and probably keep the golfers (in particular, my father-in-law) happy.

Combinatorial designs

Mathematicians refer to the solution of a problem similar to the one above as a *combinatorial design*, or simply a *design*.

DEFINITION. A **design** is a set of v *points* together with a set of subsets of size k of these points, called *blocks*, with the property that any t points lie in exactly λ blocks. Such a design with these parameters is called a $t-(v, k, \lambda)$ design.

That's a lot of variables. Let's look at an example. In our first solution to the problem, we had 16 golfers playing in groups of 4 such that every pair of golfers played together exactly once. Hence, the number of golfers $= v = 16$, the size of the groups (or blocks) $= k = 4$, and every $t = 2$ golfers play together exactly $\lambda = 1$ times. The affine plane model provided us with a $2-(16, 4, 1)$ design that solved the problem.

Based on our discussion in the last section, we now desire a $3-(16, 4, 1)$ design. That is, we want every three golfers to play together exactly once. Further, we would like to assign 4 pairwise disjoint groups to each day for 35 days. So, not only do we need to build a $3-(16, 4, 1)$ design, but we need to be able to divide the blocks of the design into 35 sets of 4 pairwise disjoint blocks each (a design with this property is called *resolvable*). It sounds like a big task, but it turns out that the design we seek was actually discovered many decades ago. One excellent source for such information is the *CRC Handbook of Combinatorial Designs* [**4**]. It is here that you can find all known values of t, v, k, and λ for which a design exists. The existence of the design we seek is due to Hanani [**5**]. However, the construction of this design relies on first finding a $3-(8, 4, 1)$ design (that is, finding an equivalent golf schedule for only 8 golfers rather than 16). Oddly enough, the construction of this smaller design also has a connection to geometry.

Projective planes

There is a close connection between affine planes and the so-called *projective planes*. Projective planes correspond to the notion of perspective. That is, from the perspective of a man standing on railroad track, the tracks seems to meet out at the horizon. Hence, parallel lines do not seem to exist. This can be laid out mathematically as follows.

DEFINITION. A **projective plane** is a set of points, together with a set of subsets of these points, called lines, such that

1. every two distinct points determine a unique line,
2. every two distinct lines meet in a unique point, and
3. there exist four points, no three of which are collinear.

Just as we did with the affine plane, we can use a vector space to model a projective plane. This time, we start with a 3-dimensional vector space V over some field $\mathcal{F}$. We take as our points the 1-dimensional subspaces of V. The 2-dimensional subspaces of V are our lines. Since two distinct 1-dimensional subspaces determine a unique 2-dimensional subspace, axiom 1 is satisfied. Similarly, two distinct 2-dimensional subspaces meet in a unique 1-dimensional subspace. Hence, axiom 2 follows. Finally, we can easily find vectors to satisfy axiom 3. For instance, we could use the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$.

For our purposes, we only need one specific projective plane. Referring to the vector space model above, it would correspond to a 3-dimensional vector space over the finite field with only 2 elements, $GF(2)$. This is probably the most famous projective plane and is more commonly known as the Fano plane. It contains 7 points, 7 lines (one of which is represented by the circle), and every line contains exactly 3 points (see FIGURE 2).

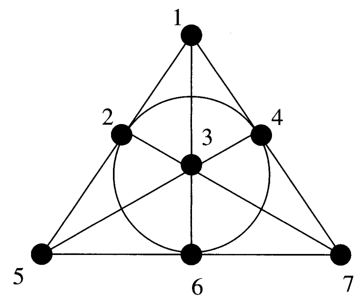

Figure 2 The Fano plane

From the Fano plane, we get a $2-(7,3,1)$ design by letting the lines of the plane represent the blocks of our design (see [**2**] for much more on this famous design). That's not quite what we want. Recall that we are looking for a design on 8 points, not 7, in order to eventually build the larger design on 16 points. We can use the Fano plane to build the design we need.

Label the points of the Fano plane with the integers 1 through 7 and consider the set of these points together with one additional point labelled 8. These will be the points of our new design. The blocks for our new design are of two types. The first type of block is a line of the Fano plane together with the extra point 8. The second type of block is any set of four points of the Fano plane such that no three of the points are collinear. Such a set of points is known as a *hyperoval.* For instance, in the labelling in FIGURE 2, note that points 1, 2, 3, and 4 form a set of 4 points, no 3 of which lie on a common line. Hence, these points form a hyperoval. Enumerating all such hyperovals and combining these with the other type of blocks defined above, we obtain the 14 blocks in TABLE 2.

TABLE 2: Blocks of the $3-(8,4,1)$ design.

1	$\{1,2,5,8\}$	$\{3,4,6,7\}$
2	$\{1,3,6,8\}$	$\{2,4,5,7\}$
3	$\{1,4,7,8\}$	$\{2,3,5,6\}$
4	$\{2,3,7,8\}$	$\{1,4,5,6\}$
5	$\{2,4,6,8\}$	$\{1,3,5,7\}$
6	$\{3,4,5,8\}$	$\{1,2,6,7\}$
7	$\{5,6,7,8\}$	$\{1,2,3,4\}$

Note that we can write the blocks in a table so that any two blocks in a row are disjoint. One can easily check that any three of our points (the points of the Fano plane, plus the additional point 8) lie together in exactly one block from TABLE 2. Hence, we have constructed a $3-(8,4,1)$ design. Moreover, we have solved the golfer's dilemma in the case when there are 8 golfers. That is, we have constructed a 7-day schedule (the rows of TABLE 2) in which every 3 golfers will play together exactly once.

A better solution

We can use the $3-(8,4,1)$ design to build the $3-(16,4,1)$ design we seek. For each row of TABLE 2, we will construct a golf schedule for 5 days, thereby giving us the 35 day schedule we need. Let $B=\{a,b,c,d\}$ be any block from TABLE 2. Then the block B constructs 10 new blocks in the manner shown in TABLE 3.

TABLE 3: 10 new blocks from the old block $\{a, b, c, d\}$.

1	$\{a,b,c,d\}$	$\{a+8,b+8,c+8,d+8\}$
2	$\{a+8,b+8,c,d\}$	$\{a,b,c+8,d+8\}$
3	$\{a+8,b,c+8,d\}$	$\{a,b+8,c,d+8\}$
4	$\{a+8,b,c,d+8\}$	$\{a,b+8,c+8,d\}$
5	$\{a,b,a+8,b+8\}$	$\{c,d,c+8,d+8\}$

So each block of the old design from the Fano plane is used to construct 10 new blocks of the design we seek. Hence, we obtain $14\cdot 10=140$ new blocks. All we need now is to partition these 140 blocks into 35 sets (representing the days) of 4 blocks each (representing the groups of golfers).

We are finally ready to construct our solution to the golfer's dilemma. We construct the 4 sets of 4 golfers each for any particular day by first choosing a row from TABLE 2, and then constructing the four associated groups using a row from TABLE 3. For instance, if we select row 4 of TABLE 2 and row 2 of TABLE 3, we obtain the four groups:

$$\{10,11,7,8\},\{9,12,5,6\},\{2,3,15,16\},\{1,4,13,14\}.$$

It is not too difficult to see that this construction gives us what we want. First note that any particular day partitions the 16 golfers into four groups of four since the groups in any row of TABLES 2 and 3 are disjoint. In addition, 3 points of the new design determine a unique block since 3 points from the $3-(8,4,1)$ design determine a unique block. This follows since we always alter an even number of entries in the original blocks to obtain the new blocks. As a result, for any given triple $\{a, b, c\}$, we can always backtrack through the tables to find the block that contains a, b, and c. This shows that we indeed have a $3-(16,4,1)$ design.

For instance, suppose we want to find the unique block containing $\{1,11,14\}$. First, we reduce the integers by subtracting 8 from any value larger than 8 and label the results as $a=1$, $b=3$, and $c=6$. Next, we look for the row in TABLE 2 containing $\{1,3,6\}$ as a subset of a block. This is row 2. Now we look for the row in TABLE 3 that will keep a unaltered, but adds 8 to b and c. This is row 4. Hence, the day corresponding to rows 2 and 4 of the two tables (respectively) has golfers 1, 11, and 14 playing together (with golfer 8).

Note that taking all possible combinations of rows of the two tables gives us the 35 day schedule we desire. Hence, we can construct a 35 day schedule for the 16 golfers such that every group of three golfers will play together in a group exactly once.

Can we go further?

Is this the best we can do? Let's think about extending our old argument. Recall that John got nervous when he played with both Bill and Howard and that three golfers uniquely determine a group. Hence, when John, Bill, and Keith play together, the fourth golfer, say Howard, is uniquely determined. Suppose that John, Bill, and Keith enjoy playing together, but do not necessarily want Howard as their fourth every time they are together. What are we saying? Essentially, *we want every possible combination of 4 players to be together exactly once*. Is it unrealistic to ask for such an extreme condition?

Let's start with some simple counting. Again, we look at everything from John's perspective. If he plays with every possible combination of 3 other golfers, then he would have to play exactly $\binom{15}{3} = 455$ times. This would certainly not be obtainable in a summer! But mathematically, it certainly seems possible and it is (see Theorem 38.1 in [**6**]). Realistically, a solution that takes this much time to complete would probably not be feasible for the average golfer. Hence, in my opinion, the solution given in the previous section is the best possible. My father-in-law seemed to like it too.

For more on projective and affine geometry and its connections to some modern problems in design theory, as well as the theory of error correcting codes and cryptography, you may want to check out *Projective Geometry* by Beutelspacher and Rosenbaum [**1**].

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50 Years Ago in the MAGAZINE

From the preface of *Theory of Functions of a Complex Variable*, Vol. 1, by Constantin Caratheodory, New York, Chelsea Publishing Company, 1954, 314 pp., $4.95, quoted as part of a posthumous review of the book in Vol. **28**, No. 2, (Nov.–Dec., 1954), 122:

> The book begins with a treatment of Inversion Geometry (geometry of circles). This subject, of such great importance for Function Theory, is taught in great detail in France, whereas in German-language and English-language universities it is usually dealt with in much too cursory a fashion. It seems to me, however, that this branch of geometry forms the best avenue of approach to the Theory of Functions; it was, after all, his knowledge of Inversion Geometry that enabled H. A. Schwarz to achieve all of his celebrated successes.