CLASSROOM CAPSULES

EDITOR

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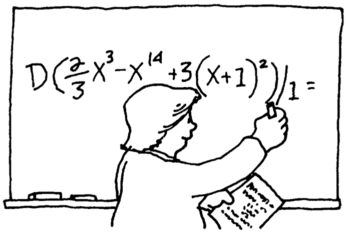

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia.

A Progression of Projectiles: Examples from Sports

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The interplay between theory and application makes elementary differential equations an attractive course. For many students, this is the time when their knowledge of calculus and linear algebra is consolidated, and this is their first chance to translate physical situations directly into mathematical equations for analysis. For a course succeeding at so many levels, it is difficult to contemplate major reform. But technology has changed the way mathematics is learned and practiced, and our courses must reflect the increased emphasis on numerical methods and nonlinear models.

The challenge is to incorporate technology without upsetting the balance of ideas that has made the course so valuable to students. At Roanoke College, we have attempted to do this by letting our topics list evolve along with the Boyce and DiPrima text [**1**], while using weekly homework projects to implement reform ideas. Our students use graphing calculators as freshmen and learn *Mathematica* as sophomores; in differential equations, we routinely use both platforms.

The projects are designed to develop a wide range of nontraditional skills. The assignments described here are the first five assignments of the term. They require graphing and low-level programming for which the graphing calculators are more than adequate. The unified nature of these projects gives students a sense of the trade-offs between realism and tractability inherent in the modeling process.

The projectile models. The projects explore projectile motion on large and small scales. As a means of making the problems real and interesting to the students, sports examples are used throughout. It is important that the students have a mental image of the process being analyzed, along with some intuition about likely results and some interest in following up the mathematical solution with interpretation. To promote this, the first day of class I bring in a variety of balls and we discuss the general problem. When an object has been "launched," what is its subsequent motion? By Newton's laws, we need to know only the forces acting on the object. These include gravity and various forces produced by friction between the projectile and the air: the drag force directed opposite to the velocity vector, and lateral forces due to spin and asymmetry of the projectile. We discuss the relative importance of these forces for a missile launch, a baseball hit, a basketball shot, and a person jumping. For most students, the concepts are familiar even if the physics is new.

The gravity-only model. We start with gravity being the only force, solving the two-dimensional model

$$x''(t) = 0, \qquad x'(0) = v_{x0}, \qquad x(0) = 0,$$
$$y''(t) = -g, \qquad y'(0) = v_{y0}, \qquad y(0) = 0.$$

The students do the integrations separately, but are asked to reconcile their results with equivalent vector equations in [**2**]. The translation part of the assignment has shown us how much help our students need with technical reading. Some simple calculations show that y is in the top 25% of its range for fully half the time of flight (see Figure 1). The students are then asked to explain why Michael Jordan seems to hang in the air.

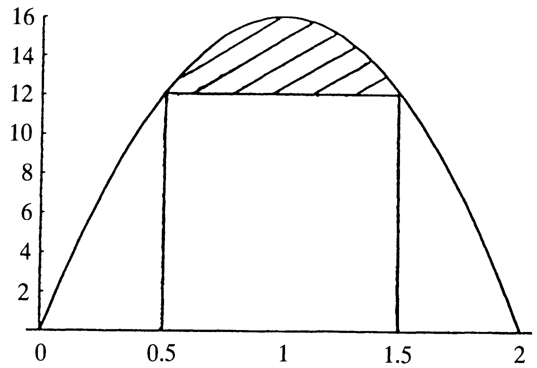

Figure 1
Half the time is spent in the top quarter of the trajectory.

The linear drag model. In the second assignment, we incorporate air resistance into the model by including a term to represent the drag force. At this point in the course, we can handle linear drag but not the more realistic quadratic drag. Students are invited to comment on whether the wrong form of drag gives a better model than no drag at all. The model is now

$$x''(t) = -c_X x'(t), \qquad x'(0) = v_{x0}, \qquad x(0) = 0$$
$$y''(t) = -c_Y y'(t) - g, \qquad y'(0) = v_{y0}, \qquad y(0) = 0$$

Again, the students solve the equations separately and compare with the dimensionless vector formulation in [**2**]: The substitutions $X = gx/v_0^2$, $Y = gy/v_0^2$, and

$T = gt/v_0^2$ ($v_{x0} = v_0 \cos\alpha$, $v_{y0} = v_0 \sin\alpha$) lead to

$$X''(T) = -\varepsilon X'(T), \qquad X'(0) = \cos\alpha, \qquad X(0) = 0$$
$$Y''(T) = -\varepsilon Y'(T) - 1, \qquad Y'(0) = \sin\alpha, \qquad Y(0) = 0$$

The students are prompted to cite the benefits of the dimensionless equations: fewer constants, and the identification of ε (equal to the ratio of initial drag to weight) as the significant quantity for a specific flight path. A comparison of the graph of this solution with the solution of the gravity-only model (Figure 2) reveals the smaller horizontal range and asymmetric path we should expect.

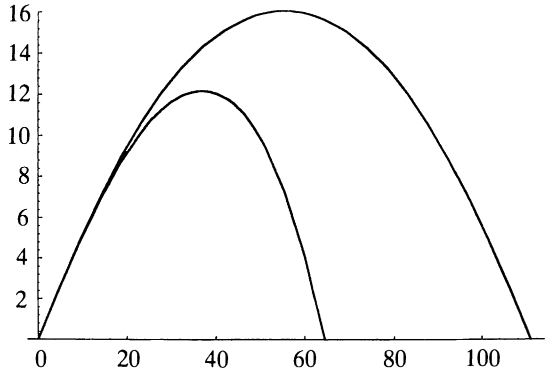

Figure 2
Comparison of linear drag and no-drag solutions.

The quadratic drag model. The model is now upgraded to include an air resistance term with drag proportional to velocity squared. This is in accordance with experimental data for most objects and speeds that would interest us [**3**]. At this stage, the students have some experience with nonlinear equations, so they are not surprised that the model is unsolvable in closed form. Most of the assignment is qualitative (we have not yet introduced numerical methods). The students follow the development in [**2**], discussing the assumptions producing the model and interpreting and solving for the terminal velocity. The text converts the model

$$x''(t) = -kx'(t)\sqrt{(x'(t))^2 + (y'(t))^2}$$
$$y''(t) = -ky'(t)\sqrt{(x'(t))^2 + (y'(t))^2} - g$$

into the model

$$\frac{dV}{d\Psi} - V\tan\Psi = \varepsilon V^3 \sec\Psi,$$

where V is a dimensionless speed and Ψ is the angle between velocity and the positive x-axis. This equation was derived by Jacob Bernoulli and solved by Gottfried Leibniz in 1696, and is the original Bernoulli-type equation [**2**]. The students solve this equation and describe in words how knowledge of V and Ψ could be used to approximate a solution curve graphically (thus previewing Euler's method).

Spin. In the fourth assignment, the effect of spin is included. Some class time with a baseball indicates how spin works. For a spinning ball moving through the air, the friction is greater on one side than on the other, resulting in a lateral force called the Magnus force [**2**]. If $\vec{\omega}$ is the angular velocity of the projectile (the vector directed along the spin axis in accord with the right-hand rule, with magnitude equal to the rate of spin in radians per second) and $\vec{v}$ is the (translational) velocity vector, the Magnus force is in the direction of the cross product $\vec{\omega} \times \vec{v}$. We look at projectiles with various spin axes and work out the directions of the corresponding Magnus forces. For simplicity, we follow Watts and Bahill [**6**] and assume that the Magnus force is a constant multiple of $\vec{\omega} \times \vec{v}$. The model is now three-dimensional:

$$\frac{d\vec{v}}{dt} = \vec{g} - k_D|\vec{v}|\vec{v} + k_M\vec{\omega} \times \vec{v}$$

The students are again asked to comment on the modeling assumptions and are asked some questions leading them to discover the dynamic nature of the drag and spin forces (which change direction in response to changes in velocity). They are given a program applying Euler's method to this model. This is their formal introduction to Euler's method, so they are asked to explain each line of the program and why the linear approximation might be expected to work (but not very well).

In the fifth assignment, the students run Euler's method for a variety of spins and velocities chosen to represent various baseball pitches. The program from assignment 4 is adapted to the platforms the students are familiar with (*Mathematica* and the students' graphing calculators). All pitches are launched from a height of 6 feet with an initial velocity of $\langle 120, 0, 0\rangle$ (the pitcher's mound is the origin, the x-axis points toward home plate, the y-axis points toward first base, and the z-axis points up). We take $k_D = 0.0025$ and $k_M = 0.005$. Some results are given below. The angular velocity vectors were chosen to represent, respectively, no spin, a sidearm curveball, a sidearm curveball with half the spin rate, a typical right-handed curveball, and an overhand fastball.

Angular velocity $\vec{\omega}$	Position when $x = 60$
$\langle 0, 0, 0\rangle$	$(60, 0, 1.557)$
$\langle 0, 0, 20\rangle$	$(60, 1.582, 1.557)$
$\langle 0, 0, 10\rangle$	$(60, 0.791, 1.557)$
$\langle -3, 14, 6\rangle$	$(60, 0.466, 0.325)$
$\langle 0, -14, 0\rangle$	$(60, 0, 2.665)$

Note that the typical fastball and curveball create a substantial variation in the height of the pitch as it reaches home plate. Also, the amount of displacement in a sidearm curve appears to be proportional to the spin rate. The students comment on these results and experiment with angular velocity vectors of their own choice.

More applications and models. Although this is as far as we take the projectile model in elementary differential equations, there are several avenues for further study.

The three-dimensional spin model given above describes a number of other projectiles of interest. An unusual result involves the motion of a golf ball. Golf clubs impart a tremendous amount of backspin to the ball, producing an upward spin force. The dominance of this lift over drag leads to the surprising result that a golf ball travels significantly farther in air than it would in a vacuum. For a (typical drive) launch angle of 11°, initial speed of 200 ft/sec, angular velocity of $\langle 0, -35, 0\rangle$, and constants $k_D = 0.0004$ and $k_M = 0.003$, the horizontal range is 827 feet, versus a horizontal range of 468 feet in a vacuum. Modeling issues include reducing k_D and k_M from the baseball values to reflect the smaller size of the golf ball. A further reduction in drag is due to the golf ball's dimples. These indentations create a turbulent wake which, contrary to most people's intuition, *reduces* the drag by a factor of 3 or 4 [**4**].

A tennis serve also lends itself nicely to analysis. For a given launch height and initial velocity, the margin of error of a serve can be investigated by finding the launch angle that causes the serve to hit the back of the service line (a distance of 60 feet) and then finding the angle that causes the serve to hit the top of the net (a distance of 39 feet at a height of 3 feet). For a "flat" serve (no spin and no drag, with $h_0 = 8$, $v_0 = 160$), the angles are approximately 5.45° and 5.91° for a margin of error of less than 0.5 degrees [**5**]; see Figure 3. A small amount of topspin can increase the margin of error to 1 degree or more, making for a much more reliable serve.

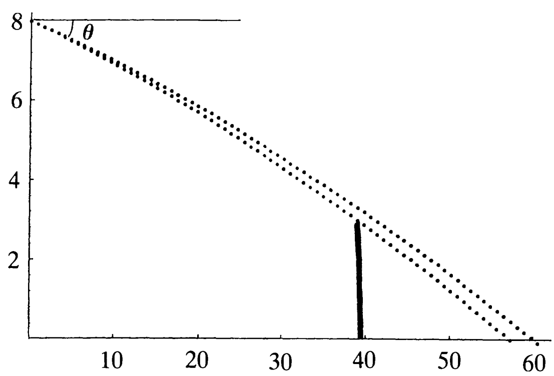

Figure 3
The margin of error of a tennis serve is less than 0.5°.

The models discussed above all neglect the asymmetries of the projectile. One situation where asymmetries play an important role is baseball's knuckleball pitch. Current research indicates that the erratic motion of the knuckleball is caused by the raised stitches of the baseball (Figure 4).

The asymmetric cross section of the ball results in a lateral force, just as the asymmetric cross section of an airplane wing results in a lift force. If the ball slowly rotates, the lateral force can change direction several times, so the knuckleball "dances." Experimentally, for a 70-mph pitch, the lateral force is well modeled by $-\frac{1}{4}\sin\theta$ lbs [**5**], where θ is an angle indicating the orientation of the baseball. For an initial angle θ_0 and angular speed ω, we have $\theta = 4\omega t + \theta_0$. The factor of 4 can be explained by noting that a great circle on a baseball will typically cross four rows of stitches separated by approximately equal intervals. The resulting period of $\pi/2\omega$ matches the observed period of the oscillating force [**5**]. The mass of a baseball is approximately $\frac{5}{512}$ slugs, so the lateral acceleration of the baseball will be $-(\frac{1}{4})(\frac{512}{5})\sin(4\omega t + \theta_0)$ ft/sec^2. Integration of $x'' = -25.6\sin(4\omega t + \theta_0)$ with

Figure 4
Stitches on a baseball.

$x(0) = x'(0) = 0$ gives a simple multiparameter model of the knuckleball. Figure 5a shows the lateral displacement x as a function of t and ω ($\theta_0 = 3\pi/2$). Note that for $\omega \geq 10$ (about 1 revolution for a 70 mph pitch), there is essentially no movement. A particularly nasty knuckleball is shown in Figure 5b (x versus t with $\omega = 4$ and $\theta_0 = 3\pi/2$). Although this pitch does not cross the center of the plate, it would be a strike (x is measured in feet). The motion of the ball depends on the

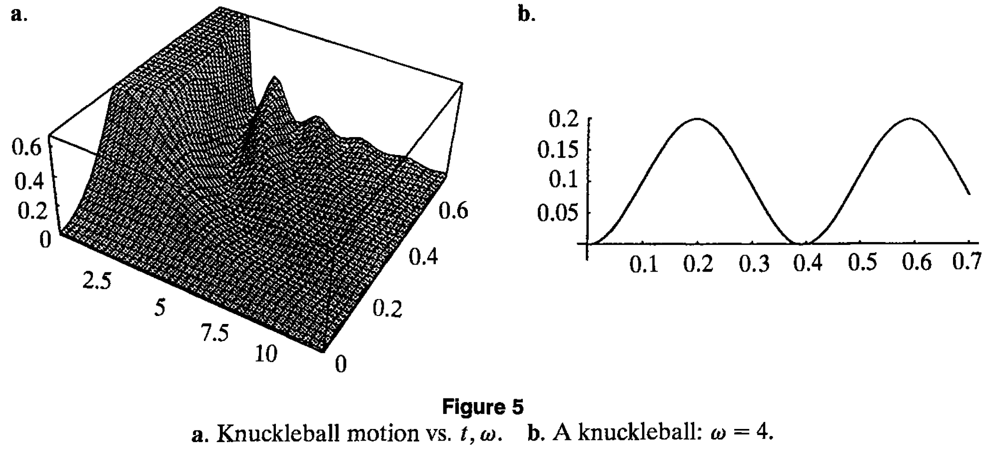

Figure 5
a. Knuckleball motion vs. t, ω. **b.** A knuckleball: $\omega = 4$.

values of ω and θ_0. Both parameters are affected by how the pitcher grips the ball. Presumably, a pitcher determines effective values by trial and error.

Conclusions. Assessment of learning is a tricky business. An undeniable conclusion is that the modeling assignments make me and my students work harder on the course. The students spend a lot of time talking among themselves and with me as they struggle with these nonstandard problems. I find the course more interesting to teach, and the students confess an increased appreciation of its usefulness. I think the exploratory nature of these assignments produces better problem solvers, but there is no hard evidence for this opinion. Since the projectile models parallel our development of solution techniques, the assignments provide a meaningful way for students to assess their growing abilities as users of differential equations.

References

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A Balloon Experiment in the Classroom

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The use of collaborative learning, group projects, in-class activities, and technology in teaching mathematics has received increased attention in the last ten years, especially in the calculus arena. In teaching differential equations (as part of the calculus course or as a separate course) one medium in which these learning styles can be integrated is that of physical experimentation in the classroom, which emphasizes the role of differential equations in modeling [**2**].

The following experiment involves a balloon, a stopwatch, and a measurement device such as a meter stick. A similar experiment is described in [**1**]. The goals of the experiment include fostering student participation, solving differential equations, dealing with parameters, comparing theory with experiment, and writing a group report. (*Note*: Handouts that discuss the derivation of the governing differential equation or the guiding questions for the write-up can be obtained from the author.)

Experiment. The experiment consists of dropping a balloon from two different heights and recording the time it takes to hit the floor. The students are directed to complete a series of steps:

A. Record the mass of the skin of the balloon. Then fill the balloon with air.
B. Drop the balloon (filled with air) from a height of 2 meters. Record the time it takes to hit the floor. Repeat this a few times. Then compute and record an average reading. Call this time T_2.