1.1

What is Geometry?

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I have put the title of my address in the form of a definite question, to which I propose to return an equally definite answer. I wish to make it quite plain from the beginning that there will be nothing in the least degree original, still less anything paradoxical or sensational, in my answer, which will be the orthodox answer of the professional mathematician.

I expect that you, as members of an association which stands half-way between the ordinary mathematical teacher and the professional mathematician in the narrower sense, will probably agree with me that I am wiser to avoid topics of what is usually called a 'pedagogical' character. I am sorry to be compelled to use the unpleasant word 'pedagogical' and I am sure that you will believe me when I say that I do not use it in any contemptuous sense, and that I am enough of a pedagogue myself to realise the very genuine interest of many 'pedagogical' questions. But I do not regard it as the business of a professional mathematician to concern himself primarily with such questions, and, even if I did, I should have very little to say about them. It has always seemed to me that in all subjects, and most of all in mathematics, questions concerning methods of teaching, whether this should come before that, and how the details of a particular chapter are best presented, however interesting they may be, are of secondary importance; and that in mathematics at all events there is one thing only of primary importance, that a teacher should make an honest attempt to understand the subject he teaches as well as he can, and should expound the truth to his pupils to the limits of their patience and capacity. In a word, I do not think it matters greatly what you teach, so long as you are really certain what it is; and I feel that you might reasonably be impatient with me, whether you agreed with me or not, if I occupied your attention for an hour and had nothing more to say to you than that. It is obviously better that I should take some definite chapter of mathematical doctrine, a chapter which is at any rate of

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the most obvious and direct educational interest, and expound it to you as clearly as I can.

It is, however, quite likely that some of you, and particularly any genuine geometer who may be present, will criticise my choice of a subject in a manner which I might find a good deal more difficult to meet. You might object that it would be reasonable enough for me to try to expound the differential calculus, or the theory of numbers, to you, because the view that I might find something of interest, to say to you about such subjects is not *prima facie* absurd; but that geometry is, after all, the business of geometers, and that I know, and you know, and I know that you know, that I am not one; and that it is useless for me to try to tell you what geometry is, because I simply do not know. And here I am afraid that we are confronted with a regrettable but quite definite cleavage of opinion. I do not claim to know any geometry, but I do claim to understand quite clearly what geometry is.

I think that this claim is in reality not quite so impertinent as it may seem. The question 'What is geometry?' is not, in the ordinary sense of the phrase, a geometrical question, and I certainly do not think it absurd to suppose that a logician, or even an analyst, may be better qualified to answer it than a geometer. There have been very bad geometers who could have answered it quite well, and very great geometers, such as Apollonius, Poncelet, Darboux, who would probably have answered it extremely badly. It is a comfort, at any rate, to reflect that my answer can hardly be worse than theirs would in all probability have been.

I propose, then, to cast doubts of this sort aside, and to proceed to answer my question to the best of my ability. There are two things, I think, which become quite clear the moment we reflect about the question seriously. In the first place, there is not one geometry, but an infinite number of geometries and the answer must to some extent be different for each of them. In the second place, the elementary geometry of schools and universities is not this or that geometry, but a most disorderly and heterogeneous collection of fragments from a dozen geometries or more. These are, or should be, platitudes, and I have no doubt that they are to some extent familiar to all of you; but it is a small minority of teachers of geometry that has envisaged such platitudes clearly and sharply, and it is probably desirable that I should expand them a little.

I begin with the second. It is obvious, first, that a great part of what is taught in schools and universities under the title of geometry is not geometry, or at any rate mathematical geometry, at all, but physics or perhaps philosophy. It is an attempt to set up some kind of ordered explanation of what has been humorously called the real world, the world of physics and sensation, of sight and hearing, heat and cold, earthquakes and eclipses; and earthquakes and eclipses are plainly not constituents of the world of mathematics.

It is dangerous to repeat truisms in public, and the particular truism which I have just stated to you is one which I have often expressed before, and which has sometimes been received in a manner very different from that which I had anticipated. But I am not speaking now to an audience of rude and simple physicists, or of philosophers dazed by centuries of Aristotelian tradition, but to one of mathematicians familiar with common mathematical ideas. I find it difficult to believe that any mathematician of the twentieth century is quite so unsophisticated as to suppose that geometry is primarily concerned with the phenomena of spatial perception, or the physical facts of the world of common sense. It is, however, perhaps unwise to take too much for granted, and I will therefore try to drive home my point by a simple illustration.

Imagine that I am giving an ordinary mathematical lecture at Oxford, let us suppose on elementary differential geometry, and that I write out the proof of a theorem on the blackboard. John Stuart Mill would have maintained that the theorem was at the best approximately true, and that the closeness of the approximation depended on the quality of the chalk; and, though Mill was a man for whom I feel in many ways a very genuine admiration, I can hardly believe that there is anybody quite so innocent as that today. I want, however, to push my illustration a stage further. Let us imagine now that a very violent dynamo, or an extremely heavy gravitating body, is suddenly introduced into the room. Einstein and Eddington tell us, and I have no doubt that they are right, that the whole geometrical fabric of the room is changed, and every detail of the pattern to which it conforms is distorted. Does common sense really tell us that my theorem is no longer true, or that the strength or weakness of the arguments by which I have established it has been in the very slightest degree affected? Yet that is the glaring and intolerable paradox to which anyone is committed who supports the old-fashioned view that geometry is 'the science of space'.

The simple view, then – the view which I will call for shortness the view of common sense, though there is uncommonly little common sense about it – the view that geometry is the science which tells us the facts about the space of physics and sensation, is one which will not stand a moment's critical examination; and this, of course, was plain enough before Einstein, though it is Einstein who, by enabling us to exhibit its paradoxes in so crude a form, has finally completed the demonstration. The philosophers, of course, have tried to restate the view of common sense in a more sophisticated form. Geometry, they have explained to us, tells us, not exactly the facts of physical or perceptual space, but certain general laws to which all spatial perception must conform. Philosophers have been singularly unhappy in their excursions into mathematics, and this is no exception. It is, as usual, an attempt to restrict the liberty of mathematicians, by proving that it is impossible for them to think except in some particular way; and the history of mathematics shows conclusively that mathematicians will never accept the tyranny of any philosopher. The moment a philosopher has demonstrated the impossibility of any mode of thought, some rebellious mathematician will employ it with unconquerable energy and conspicuous success. No sooner was the apodeictic certainty of Euclid firmly established, than the non-Euclidean geometries were constructed; no sooner were the inherent contradictions of the infinite finally exposed, than Cantor erected a coherent theory. I do not think, then, that we need trouble ourselves with the views of the philosophers concerning geometry. They are, indeed, of much less interest than those of the man in the street, which do possess some interest, since there are valid reasons for supposing that others may share them.

It will be more profitable to leave the philosophers alone, and to consider what the mathematicians themselves have to say. We shall then have reasonable hope of making some substantial progress, since mathematicians, or those of them who are at all interested in the logic of mathematics, hold fairly definite views, and views which are in tolerable agreement, concerning this question of the relation of geometry to the external world. The views of the mathematicians are also much more modest than those which the philosophers have tried to impose upon them.

A geometry like any other mathematical theory, is essentially a map or scheme. It is a picture, and a picture, naturally, of *something*; and as to what that something is opinions do and well may differ widely. Some will say that, it is a picture of something in our minds, or evolved from them or constructed by them, while others, like myself, will be more disposed to say that it is a picture of some independent reality outside them; and personally I do not think it matters very much which type of view you may prefer to adopt. What is much more important and much clearer is this, that there is one thing at any rate of which a geometry is not a picture, and that that is the so-called real world. About this, I think that almost all modern mathematicians would agree.

This is only common mathematical orthodoxy, but it is an orthodoxy which outsiders very frequently misunderstand or misrepresent. I need hardly say that it does not mean that mathematicians regard the world of physical reality as uninteresting or unimportant. That would be on a par with the view that mathematicians are peculiarly absent-minded, always lose at bridge, and are habitually unfortunate in their investments. Still less does it mean that they regard as uninteresting or unimportant the contribution which mathematics can make to the study of the real world. The Ordnance Survey suggests to me that Waterloo Station, and Piccadilly Circus, and Hyde Park Corner lie roughly in a straight line. That is a geometrical statement about reality, and it enables me to catch my train at Paddington. Einstein is more daring, and issues his orders to the stars, and the stars halt in their courses to obey him. Einstein, and the Ordnance Survey, and even I, can all of us, armed with our mathematics, put forward suggestions concerning the structure of physical reality, and our suggestions will continually prove to be not merely interesting, but of the most direct and practical importance. We can point to this or that mathematical model, Euclidean or Lobatschewskian or Einsteinian geometry, and suggest that perhaps the structure of the universe resembles it, or can be correlated with it in one way or another; that that is a possibility at any rate which the physicists may find it worth their while to consider. We can offer these suggestions, but, when we have offered them, our function as mathematicians is discharged. We cannot, do not profess to, and do not wish to *prove* anything whatsoever. There is not, and cannot be, any question of a mathematician proving any thing about the physical world; there is one way only in which we possibly discern its structure, that is to say the laboratory method, the method of direct observation of the facts.

I will venture here on an illustration which I have used before. If one of you were to tell me that there are three dimensions in this room, but five for Southampton Row, I should not believe him. I would not even suggest that we should adjourn our discussion and go outside to see. The assertion would of course, be one of an exceedingly complicated character, and a very painstaking analysis might prove necessary before we were quite certain what it meant. However, I could attach a definite meaning to it. I should understand it to imply that, owing to particularities in the geography of London which had up to the present escaped my attention, the common three-dimensional model, sufficient for our purposes in here, becomes inadequate when we pass out into the street. And, however sceptical I might feel about such a theory, I should certainly not be so foolish as to advance mathematical arguments against it, for the all-sufficient reason that I am quite certain that there are none. I should be sceptical, not as a geometer but as a citizen of London, not because I am a mathematician, but in spite of it; and, indeed, I am sure that, if you appealed from me to the nearest policeman, you would find him not less but far more obstinately sceptical than me.

I must pass on, however, to what is really the proper subject matter of my address. Geometries, I will ask you to agree provisionally, are *models*, and models of something which, whatever it may be in the last analysis, we may allow for our present purposes to be described as mathematical reality. The question which we have now to consider is that of the nature of these models; and the characteristics which distinguish one from another; and there is one great class of geometries for which the answer is immediate and easy, namely, that of the *analytical* geometries.

An analytical geometry, whether of one, two, three, four, or n dimensions, whether real or complex, projective or metrical, Euclidean or non-Euclidean, and it may, of course, be any of these, is a branch of analysis concerned with the properties of certain sets or classes of sets of numbers. I will take the simplest example, the two-dimensional Cartesian geometry which resembles very closely, though it is by no means the same as, the elementary 'analytical geometry' taught in schools. I will call it, as I usually call it in lectures, *Common Cartesian Geometry*.

In Common Cartesian Geometry, a *point* is, by definition, a pair of real numbers (x, y), which we call its *coordinates*. A line is, again by definition, a certain class of points, viz. those which satisfy a linear relation $ax + by + c = 0$, where a, b, c are real numbers and a and b are not both zero. The relation itself is called the *equation* of the line. If the coordinates of a point satisfy the equation of a line, the line is said to *pass through* the point, and the point to *lie on* the line. And that is the end of Common Cartesian Geometry, in so far as it is projective, that is to say in so far as it does not use the so-called metrical notions of distance and angle, and in so far as it is concerned only with equations of the first degree. What remains is just algebraical deduction from the definitions.

Common Cartesian Geometry, as I have defined it, is a very simple and not a very interesting subject. It gains a great deal in interest, as you will readily imagine, when 'metrical' concepts are introduced. We define the *distance* of two points (x_1, y_1) and (x_2, y_2) by the usual formula

$$d = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}},$$

and the *angle* between two lines by another common formula, which I need not repeat. We have still, however, only to explore the algebraical consequences of our definitions, and no new point of principle arises, so that I can illustrate what I want to say quite adequately from the projective and linear system. This system, trivial as it is, has certain features to which I wish to call your attention as characteristic of analytical geometries in general.

The first feature is this, that a point in Common Cartesian Geometry is *a definite thing*. This is so in all analytical geometries. Thus in any system of two-dimensional and homogeneous analytical geometry a point is a class of triads (x, y, z), those triads being classified together whose coordinates are proportional, and in the geometry of Einstein a point is a set of four numbers (x, y, z, t). This is a very obvious observation, but it is of fundamental importance, since it marks the most essential difference between analytical geometries and 'pure' geometries, in which, as we shall see, a point is not a definite entity at all.

The next point which I ask you to observe is the absence of *axioms*. There are no axioms in any analytical geometry. An analytical geometry consists entirely of *definitions* and *theorems*; and this is only natural, since the object of axioms is, as we shall see, merely to limit our subject matter, and in an analytical geometry our subject matter is known.

It is most important to realise clearly that, in different geometrical systems, propositions verbally identical may occupy entirely different positions. What is an axiom in one system may be a definition in another, a true theorem in a third, and a false theorem in a fourth. You are accustomed, for example, to *proving* that the equation of a straight line is of the first degree, and I am not suggesting that the 'proof' to which you are accustomed is meaningless, trivial, or false. You profess to be proving a theorem, and you are, in fact, genuinely proving something, though it might take us some time to ascertain exactly what it is. There is one thing, however, that is quite plain, and that is that the something which you are proving is not a theorem of analytical geometry, for your supposed theorem is, as a proposition of analytical geometry, not a theorem at all but the definition of a straight line.

Let us take another simple illustration, the 'parallel postulate' of Euclid. *If L is a line, and P is point which does not lie on L, then there is one and only one line through P which has no point in common with L.* This, in school geometry, is sometimes called an 'axiom' and sometimes, I suppose, an 'experimental fact'. It cannot be either of these in analytical geometry, where there are neither axioms nor experimental facts, and it is obviously not a definition. It is, in fact, a theorem, which in Common Cartesian Geometry is true, though in other systems it may be false; and it is a theorem which any schoolboy can prove. It is the algebraical theorem that, given an equation $ax + by + c = 0$, and a pair of numbers, x_0, y_0, which do not satisfy this equation, then it is possible to find numbers A, B, C, such that

$$Ax_0 + By_0 + C = 0 \qquad (1)$$

and the equations

$$ax + by + c = 0, \quad Ax + By + C = 0 \qquad (2)$$

are inconsistent with one another; and that the ratios $A : B : C$ are determined uniquely by these conditions.

These are the characteristics of Common Cartesian Geometry which it is most essential for us to observe at the moment. There are others which I should like to say something about if I had time. There is no infinite and no imaginary in this geometry; there are imaginaries, naturally, only in complex systems, and infinites in homogeneous systems. Further, the principle of duality is untrue. All these topics call for comment; and I should have liked particularly to say something on the subject of the geometrical infinite, since the tragical misunderstandings which have beset many writers of text-books of analytical geometry, and which have generated such appalling confusion in the minds of university students, are misunderstandings for which writers like myself of text-books on analysis have been largely though innocently responsible. The geometrical infinite, however, is a subject which would demand at least a lecture to itself. Apart from this, there is nothing in analytical geometry which presents any logical difficulty whatever, and I may pass to the slightly more delicate topic of pure geometry.

The nature of a system of pure geometry, such as the ordinary projective system, is most easily elucidated, I think, by contrast with analytical systems. The contrasts, which I have made by implication already, are sharp and striking, and when once they have been clearly observed the road to the understanding of the subject is open. I observed, first, that the points and lines of analytical geometry were *definite objects*, such as the pair of numbers (2, 3). Secondly, I observed that there were no *axioms* in an analytical geometry, which consists of definitions and theorems only; and that it is the definitions which differentiate one system of analytical geometry from another. The business of an analytical geometer is, in short, to investigate the properties of *particular systems of things*. The standpoint of a pure geometer is entirely different. He is not, except for incidental and subsidiary purposes, concerned with particular things at all. His function is always to consider *all things which possess certain properties*, and otherwise to be strictly indifferent to what they are. His 'points' and 'lines' are neither spatial objects, nor sets of numbers, nor this nor that system of entities, but *any* system of entities which are subject to a certain set of logical relations. The particular system of relations which he studies is that which is expressed by the *axioms* of his geometry. It is the axioms only which really matter; it is they which discriminate systems, and the definitions play an altogether subsidiary part.

Suppose, for example, if I may take a frivolous illustration, that a pure geometer and an analytical geometer were to go together to the zoo. The analytical geometer might be interested in tigers, in their colour, their stripes, and in the fact that they eat meat. A point, he would say, is by definition a tiger, and the central theorems of my geometry are that 'points are yellow,' that 'points are striped,' and above all that 'points eat meat.' The pure geometer would reply that he was quite indifferent to tigers, except in so far as they possessed the properties of being yellow and striped; that *anything* yellow and striped was a point to him; that 'points are yellow' and 'points are striped' were the *axioms* of his geometry, and that all he wanted to know was whether 'points eat meat' is a logical deduction from them.

You will, in fact, find, if you consult any standard work on pure geometry, such as Hilbert's *Grundlagen* or Veblen and Young's *Projective Geometry*, that a pure geometer begins somewhat as follows. We consider a system S of objects $A, B, C, \ldots$. We call these objects *points*, and their aggregate *space*; the *plane*, I may say, if I confine myself for simplicity to geometries of two dimensions. From the complete system S which constitutes space we pick out certain partial aggregates $L, M, N, \ldots$, which we call *lines*. If a point A belongs to the particular partial aggregate L, we say that A *lies on* L and that L *passes through* A. These are the *definitions*, and you will observe the quite subsidiary part they play. They are, in fact, purely verbal, and common to all systems; and they do not indicate or imply any special property whatever of the objects which they are said to define, which are indeed often called the *indefinables* of the geometry. The function of the definitions, in fact, is merely to point to the indefinables.

The serious business of the geometry begins when the axioms are introduced. We suppose next that our points and lines are subject to certain logical relations. These suppositions are assumptions, and we call them axioms. To construct a geometry is to state a system of axioms and to deduce all possible consequences from them.

Let us take an actual example. I select the following system of axioms:

Axiom 1. *There are just three different points.*
Axiom 2. *No line contains more than two points.*
Axiom 3. *There is a line through any two points.*

These axioms are consistent with one another, for it is easy to construct a system of objects which satisfy them. We might, for example, take the numbers 1, 2, 3 as our points and the pairs of numbers 2 3, 3 1, 1 2 as our lines, in which case all our axioms are obviously satisfied. Further, the axioms are independent of one another. If the numbers 1, 2, 3 were still our points, but the pairs 2 3, 3 1 alone, and not the pair 1 2, were taken as lines, then the first and second axioms would be satisfied but not the third, and it naturally follows that Axiom 3 is incapable of deductions from the other two. You will have no difficulty in proving in a similar manner, if you care to do so, that each of the three axioms is logically independent of the others. I do not profess to have stated the axioms in the best form possible, but at any rate they are consistent and independent.

It is easy to deduce from our axioms:

Theorem 1. *There are just three lines.*
Theorem 2. *There are just two lines through any point.*

The state of affairs in this geometry is, in short, that suggested by a figure consisting of three points on a blackboard and three lines joining them in pairs. With this, our geometry appears to be exhausted.

The geometry which I have constructed is not an interesting system, since it has no particular application and virtually no content. For our present purpose, however, that is an advantage, as it makes it possible for me to exhibit the system to you in its entirety. However little interest, it may possess, it is a perfectly fair specimen of a pure geometry. All systems of pure geometry, projective geometry, metrical geometry Euclidean or non-Euclidean, are constructed in just this way. They are usually very much more complicated, for you must naturally be prepared to sacrifice simplicity to some extent if you wish to be interesting; but their differences from my trivial geometry are differences not at all of principle or of method, but merely of richness of content and variety of application.

I have now given to you the substance of the orthodox answer to the question which I started by asking. I might expand it indefinitely in detail, but I should add nothing essentially new. Geometry is a collection of logical systems. The number of systems is infinite, and any of you can invent as many new systems as you please; I have myself, with the aid of a few pupils, constructed seven or eight in the course of an hour. There are two kinds of systems, analytical geometries and pure geometries. An analytical geometry attaches the usual geometrical vocabulary to more or less complicated systems of numbers, and investigates their properties by means of the ordinary machinery of algebra and analysis. A pure geometry, on the other hand, considers all possible fields of certain logical relations, and explores their connections without reference to the nature of the objects among which they hold.

I said when I started that I did not propose to offer any very definite suggestions about the teaching of mathematics; but I should like to conclude with a few words about some of the practical problems with which members of this association are primarily concerned. It should be obvious to you by now, I think, that school geometry is, as I stated early in my address, not a well-defined subject, a rational exposition of a particular geometrical system, but a collection of miscellaneous scraps, a selection of airs from different pieces, strung together in the manner which experience shows to be the most enlivening. It would be very easy for me to illustrate my thesis by examining a few passages from current text-books of geometry. What is taught as projective geometry, for example, is not projective geometry, and makes very little pretence of being so; since it is based quite frankly on ratios of lengths and other obviously metrical concepts. Indeed, so far as I know, no English book on projective geometry proper exists, except Mathews' *Projective Geometry*, Dr. Whitehead's tract, and parts of Prof. Baker's treatise. On the other hand, a great deal of what is taught as analytical geometry is not analytical geometry, but an attempt to apply the methods of analytical geometry in other fields, partly to some rough kind of physical geometry supposed to be given intuitively, partly to some system of hybrid pure geometry of which some previous knowledge is assumed. But I must not enter into detail, since detail would mean criticism, and criticism of particular books and particular passages, which I have no time for, and am in any case anxious to avoid.

It is not my object now to offer criticisms of the present methods of geometrical teaching. There are a good many very obvious criticisms suggested by the doctrines which I have tried to explain to you, but I recognise that most of these criticisms would be to a very great extent unfair. It is obvious that the teaching of geometry must be based on what is at best a very illogical compromise, and I am prepared to believe that the compromise evolved by experience, and applied by people who know a good deal more about the practical necessity of compromise than I do, is in substance as reasonable a compromise as the difficulties of the problem permit. My object, so far, has been one not of criticism but of explanation.

I do propose, however, to conclude with one word of criticism, directed only to those of you whose pupils are comparatively able and comparatively mature. There is no doubt that the standard of teaching of analysis has improved out of all knowledge during the last twenty years. The elements of the calculus, even the elements of what foreign mathematicians call algebraical analysis, are taught in a manner with which I personally have comparatively little fault to find. The stupid old superstition that falsehood is always easy and attractive, the truth inevitably repulsive and dull, is almost dead, and it is no longer supposed that ignorance of analysis is in itself a proof either of superior intelligence, or high moral character, or profound geometrical or physical intuition. The teaching of higher geometry does not seem to me to have advanced in the same degree.

I think that it is time that teachers of geometry became a little more ambitious. Geometry in its highest developments may be, for all I know, a more difficult subject than analysis; it is not for me as an analyst to deny it. But what may be true enough of the theory of deformation of surfaces, or of algebraical curves in space, is not even plausible of the elements of higher geometry. Those stages of the subject are surely very much easier than the corresponding stages of analysis. There is something hard and prickly about the basic difficulties of analysis, definite stages on the road where definite types of mind seem to come to an inevitable halt. The difficulties of geometry seem to me a little softer and vaguer; knowledge and general intelligence will carry a student appreciably further on the way. And, if this is so, it seems to me regrettable that students are not given the opportunity, while still at school, of learning a good deal more about the real subject matter out of which modern geometrical systems are built. It is probably easier, and certainly vastly more instructive than a, great deal of what they are actually taught. Anyone who can investigate properties of six or eight points on a conic is capable of understanding what projective geometry is. Anyone who has the faintest hope of a scholarship at Oxford or Cambridge could learn the nature of an axiom, and how a system of axioms may be shown to be consistent with, or independent of, one another. And anyone who can be taught to project two arbitrary points into the circular points at infinity could learn, what he certainly does not learn at present, to attach some sort of definite meaning to the process he performs. Small as my own knowledge of geometry is, and slight as are my qualifications for teaching it to anybody, I have not yet encountered the student who finds difficulty with such ideas when once they are put before him clearly. I am well aware of the very great services which the Association has rendered in the improvement of geometrical teaching. I think that it might well now concentrate its efforts on a general endeavour to widen the horizon of knowledge, recognising, as regards niceties of logic, sequence, and exposition, that the elementary geometry of schools is a fundamentally and inevitably illogical subject, about whose details agreement can never be reached.