vectors precisely the velocity vectors of β together with $\Omega'(0) = (1,0)$. Finally, if β has length L, then Ω has length $2L$, and if we parametrize Ω using arc length then we obtain a path Ω_0 with $\Omega_0(0) = (0,0), \Omega_0(2L) = (1,0)$, and Ω_0 has unit speed except at time 0 at which time its speed is $1/2L$.

We conclude with a two-dimensional version of L'Hôpital's Rule. (Perhaps this is appropriate as L'Hôpital's Rule seems to be one of the more popular applications of Cauchy's Generalized Mean Value Theorem.)

COROLLARY 3. *Suppose that* $\Omega(t) = (x(t), y(t))$ *is a differentiable, nonstop path defined on the closed interval* $[a,b]$. *If* Ω *is one-to-one on some neighborhood of* a, *and* $\lim \Omega(t) = (0,0)$ *as t approaches* a^+, *then as t approaches* a^+, $\lim \Omega(t)/\|\Omega(t)\| = \lim \Omega'(t)/\|\Omega'(t)\|$, *assuming the latter limit exists.*

One interpretation of this is that if a particle starts at the origin in the plane, then, under reasonable conditions, its "position directions" are closely approximated by its "velocity directions." Another interpretation, which is perhaps even more intuitive, is that if a particle has constant mass and zero initial velocity, then (under suitable conditions) its unit velocity vectors near the start are approximately its unit acceleration vectors (i.e., its force direction vectors). In other words, if a particle is at rest, then the direction it goes is pretty much the direction you push it. I guess the only real surprise here is that we are not even assuming that the force is continuous.

The proof of this corollary is identical to the usual proof of L'Hôpital's Rule given the Generalized Mean Value Theorem and is omitted.

Disks and Shells Revisited

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It is a common practice in calculus courses to use the definite integral to *define* the area between the graph of a function and the x-axis (see, e.g., [**2**, p. 252], [**3**, p. 221], and [**4**, p. 238]). Soon after, the student is taught two methods to calculate volumes of solids of revolution—the disk method and the shell method—usually with no mention of how *volume* is defined. Most calculus books follow the introduction of disks and shells with several examples in which it is shown that both methods of calculating the volume yield the same answer. The alert student is sure to wonder whether this is always the case.

The equivalence of the disk and shell methods was proven in [**1**] using integration by parts. We present here a different approach, one that uses only elementary ideas and illustrates an important proof technique.

THEOREM. *Let* $f(x)$ *be a continuous, invertible function on the interval* $[a,b]$, *where* $a \geqslant 0$. *Suppose the region bounded by* $y = f(b)$, $x = a$, *and the graph of* $f(x)$ *is rotated about the y-axis. Then the values of the volume of the resulting solid* *obtained by the disk and shell methods are equal. That is,*

$$\int_{f(a)}^{f(b)}\left(\pi\left[f^{-1}(y)\right]^2-\pi a^2\right)dy=\int_a^b 2\pi x[f(b)-f(x)]\,dx.$$

The Ingredients. Early in most calculus curricula, Rolle's Theorem is used to prove the following principle, which is then applied repeatedly.

PRINCIPLE. *If $f(x)$ and $g(x)$ are two functions that satisfy:*
(a) *$f(x)$ and $g(x)$ are differentiable on an interval $[a, b]$,*
(b) *$f'(x) = g'(x)$ for all $x \in [a, b]$, and*
(c) *$f(c) = g(c)$ for one point $c \in [a, b]$,*
then $f(x) = g(x)$ for all $x \in [a, b]$.

Informally, this says that two functions are equal if they are equal at *one* point and have identical derivatives. Although this principle is surprisingly simple, and easily absorbed by students, it has numerous applications and reappears often. Emphasizing this principle helps students see the similarity between proofs that otherwise seem unrelated.

The other ingredients of the proof of our theorem are the Fundamental Theorem of Calculus, the chain rule, and the product rule.

Proof. Let $t \in [a, b]$. We define two functions of t as follows:

$$V(t)=\int_{f(a)}^{f(t)}\left(\pi\left[f^{-1}(y)\right]^2-\pi a^2\right)dy$$

and

$$W(t)=\int_a^t 2\pi x[f(t)-f(x)]\,dx.$$

We need to prove that $V(b) = W(b)$, as these are the two volumes in the theorem. It is easy to show that $V(t) = W(t)$ for all $t \in [a, b]$ by applying the principle given above.

First simplify $W(t)$:

$$W(t)=2\pi f(t)\int_a^t x\,dx-2\pi\int_a^t xf(x)\,dx.$$

Now, by the Fundamental Theorem, both $V(t)$ and $W(t)$ are differentiable on the interval $[a, b]$. Furthermore, $V'(t) = (\pi[f^{-1}(f(t))]^2 - \pi a^2)f'(t) = \pi(t^2 - a^2)f'(t)$, and

$$W'(t)=2\pi f'(t)\int_a^t x\,dx+[2\pi f(t)t-2\pi t f(t)]$$

$$=2\pi f'(t)\left[\frac{t^2-a^2}{2}\right]=\pi(t^2-a^2)f'(t).$$

Thus, $W'(t) = V'(t)$ for all $t \in [a, b]$. It remains only to observe that $V(a) = W(a) = 0$, by a fundamental property of integrals. $\square$

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L'Hôpital's Rule Via Integration

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In elementary calculus texts L'Hôpital's rule is usually proven only for the case $0/0$, $x \to x_0$ (finite), by applying the Cauchy mean value theorem. Extension to $x \to \infty$ is then accomplished by replacing x with $1/x$. Verification of the rule for the ∞/∞ indeterminate form is regarded as too difficult and may be discussed in an exercise, an appendix, or not at all. In this note we give a proof for the ∞/∞ case that does not make use of the Cauchy mean value theorem. Instead, we require that the functions have continuous derivatives and take advantage of the order properties of the definite integral. The argument adapts nicely to the case $0/0$ as well.

L'HÔPITAL'S RULE. ∞ / ∞. *Let f and g have continuous derivatives with* $g'(x) \neq 0$. *If* $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} f'(x)/g'(x) = L$, *then* $\lim_{x\to\infty} f(x)/g(x) = L$ *also.*

Proof. We assume that L is finite; the other case can be handled in a similar fashion. The limit hypothesis on g allows us to assume that it is a positive function. Moreover, since g' is continuous and nonvanishing it too must be positive.

Let ε be some positive number. Choose M so that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon$$

whenever $x > M$. Since $g'(x)$ is positive we have

$$|f'(x) - Lg'(x)| < \varepsilon g'(x),$$

so that

$$\left|\int_a^b [f'(x) - Lg'(x)]\,dx\right| \leqslant \int_a^b |f'(x) - Lg'(x)|\,dx < \int_a^b \varepsilon g'(x)\,dx$$

whenever $M < a < b$. Therefore, for such a and b,

$$|f(b) - f(a) - L[g(b) - g(a)]| < \varepsilon[g(b) - g(a)]. \qquad (*)$$

Dividing through by the positive number $g(b)$ we obtain

$$\left|\frac{f(b)}{g(b)} - \frac{f(a)}{g(b)} - L\left[1 - \frac{g(a)}{g(b)}\right]\right| < \varepsilon\left[1 - \frac{g(a)}{g(b)}\right] < \varepsilon.$$