Math 1b Final Exam – fall '11 Name_________________________

Directions: Write all responses on separate paper. Show all work for credit. No calculators.

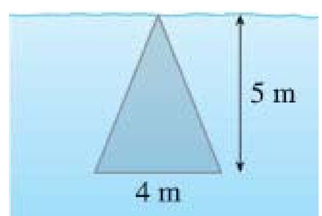
1. Consider $\int_0^{\frac{4\pi}{3}} \cos x\, dx$
   a. Simplify the approximation to the integral using the trapezoidal method with $n = 4$.
   b. Simplify the approximation to the integral using the midpoint method with $n = 4$.
   c. Simplify the approximation to the integral using the Simpson's method with $2n = 8$.
   d. Which of the above estimates is an/are overestimate(s)?

2. Consider $I = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$
   a. Use integration by parts to derive the reduction formula $I = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$
   Hint: use $u = \cos^{n-1} x$ and $dv = \cos x\, dx$.
   b. Use the reduction formula to evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$.

3. Consider the region bounded by $\cos x \le y \le \cos^2 x$ where $x \in \left[0, \frac{\pi}{2}\right]$.
   Set up and evaluate integrals to compute each of the following:
   a. The volume of revolution when this region is revolved about the x-axis, using discs.
   b. The volume of revolution when this region is revolved about the y-axis, using shells.

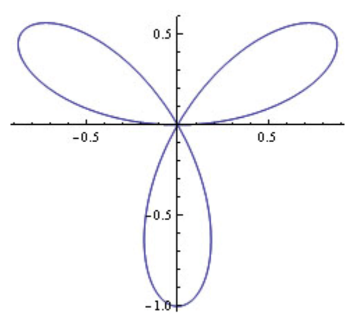
4. Find the surface area of the volume formed by revolving $y = 1 - x^2$ for $0 \le x \le 1$, about the y–axis.

5. Set up an integral to compute the total fluid force on one side of the vertical plate submerged in water as shown at right. You don't need to evaluate the integral.
   Hint: recall the element of force is
   = force density * depth * element of surface area



6. Find the arc length of the curve given by the parametric equations $x = 1 + t^2,\ y = 2 + t^3$ between the points (2, 3) and (5, 10) in the xy-plane.

7. Find the area enclosed by the curve $r = \sin(3\theta)$.
   The graph of this polar function is shown at right.
   Hint: Recall that the form of a polar integral is $\int_a^b \frac{r^2}{2}\, d\theta$



8. Expand $f(x) = \sqrt[3]{1 + x^2}$ as a binomial series to obtain a 6th degree approximating polynomial.

9. Find the radius of convergence and interval of convergence of the power series, $f(x) = \sum_{n=0}^{\infty} \frac{(2x-3)^n}{n!}$.

10. Simplify the fifth degree Taylor polynomial for $f(x) = \sin 2x$ expanded about $a = \frac{\pi}{8}$.

Math 1b Final Exam Solutions – fall '11

1. Consider $\int_0^{\frac{4\pi}{3}} \cos x\, dx$
   a. Simplify the approximation to the integral using the trapezoidal method with $n = 4$.
   SOLN: $\Delta x = \frac{\frac{4\pi}{3}-0}{4} = \frac{\pi}{3}$. So the trapezoidal method yields
   $$\int_0^{\frac{4\pi}{3}} \cos x\, dx \approx \frac{\pi}{3}\left(\frac{\cos 0}{2} + \cos\frac{\pi}{3} + \cos\frac{2\pi}{3} + \cos\pi + \frac{\cos\left(\frac{4\pi}{3}\right)}{2}\right) = \frac{\pi}{3}\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2} - 1 - \frac{1}{4}\right) = -\frac{\pi}{4}$$
   b. Simplify the approximation to the integral using the midpoint method with $n = 4$.
   $$\int_0^{\frac{4\pi}{3}} \cos x\, dx \approx \frac{\pi}{3}\left(\cos\frac{\pi}{6} + \cos\frac{\pi}{2} + \cos\frac{5\pi}{6} + \cos\left(\frac{7\pi}{6}\right)\right) = \frac{\pi}{3}\left(\frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = -\frac{\pi\sqrt{3}}{6}$$
   c. Simplify the approximation to the integral using the Simpson's method with $2n = 8$.
   $$\frac{\pi}{18}\left(\cos 0 + 4\cos\frac{\pi}{6} + 2\cos\frac{\pi}{3} + 4\cos\frac{\pi}{2} + 2\cos\frac{2\pi}{3} + 4\cos\frac{5\pi}{6} + 2\cos\pi + 4\cos\left(\frac{7\pi}{6}\right) + \cos\frac{4\pi}{3}\right)$$
   $$= \frac{\pi}{18}\left(1 + 2\sqrt{3} + 1 + 0 - 1 - 2\sqrt{3} - 2 - 2\sqrt{3} - \frac{1}{2}\right) = -\frac{\pi}{18}\left(\frac{3 + 4\sqrt{3}}{2}\right)$$
   Note that $\frac{\left(-\frac{\pi}{4}+2\left(-\frac{\pi\sqrt{3}}{6}\right)\right)}{3}\frac{12}{12} = -\frac{\pi(3+4\sqrt{3})}{36}$ confirms that Simpson's approximation is the weighted average of trapezoid and midpoint approximations.
   d. Which of the above estimates is an/are overestimate(s)?
   The true value of the integral is $\int_0^{\frac{4\pi}{3}} \cos x\, dx = \sin x|_0^{4\pi/3} = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \approx -0.866$
   Now $-\frac{\pi}{4} \approx -\frac{3.14159}{4} \approx -0.7854$ and $-\frac{\pi\sqrt{3}}{6} \approx -\frac{3.14159}{6} \cdot 1.732 \approx -0.5236 \cdot 1.732 \approx -0.907$
   This means the Simpson approximation is $\approx -\frac{(0.7854+2(0.907))}{3} = -\frac{2.5994}{3} \approx -0.8665$ so we have midpoint < Simpson < true value < trapezoid, though the Simpson approximation is so close to the true value, it's hard to distinguish without a calculator. Note that the part of the integral that doesn't cancel out is the part between π and $4\pi/3$, where the cosine curve is concave up. Thus the midpoint will be an underestimate and the trapezoid an over estimate.
2. Consider $I = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$
   a. Use integration by parts to derive the reduction formula $I = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$
   Hint: use $u = \cos^{n-1} x$ and $dv = \cos x\, dx$.
   SOLN: With $\begin{pmatrix} u = \cos^{n-1} x & dv = \cos x\, dx \\ du = -(n-1)\cos^{n-2} x \sin x\, dx & v = \sin x \end{pmatrix}$, we have
   $$I = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \cos^{n-1} x \sin x|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x\, dx$$
   $$= -(n-1)\left(I - \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx\right)$$
   We can solve this equation for I: $nI = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$ or $I = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$

   b. Use the reduction formula to evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$.
   SOLN: $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2} \cos x\, dx = \frac{16}{35}$

3. Consider the region bounded by $\cos x \le y \le \cos^2 x$ where $x \in \left[0, \frac{\pi}{2}\right]$.
Set up and evaluate integrals to compute each of the following:
   a. The volume of revolution when this region is revolved about the *x*-axis, using discs.
   SOLN: $\pi \int R^2 - r^2 dx = \pi \int_0^{\frac{\pi}{2}} \cos^2 x - \cos^4 x\, dx = \pi\left(\frac{1}{2}\frac{\pi}{2} - \frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right) = \frac{\pi^2}{16}$
   b. The volume of revolution when this region is revolved about the *y*-axis, using shells.
   SOLN: $2\pi \int rh dx = 2\pi \int_0^{\frac{\pi}{2}} x(\cos x - \cos^2 x)\, dx = \begin{pmatrix} u = x & dv = \cos x - \cos^2 x\, dx \\ du = dx & v = \sin x - \frac{x}{2} - \frac{\sin 2x}{4} \end{pmatrix}$

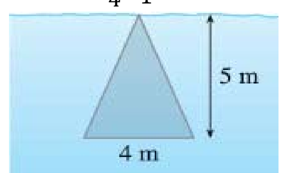
$$2\pi\left(x \sin x - \frac{x^2}{2} - \frac{x \sin 2x}{4}\,\Bigg|_0^{\pi/2}\right) - 2\pi \int_0^{\pi/2} \sin x - \frac{x}{2} - \frac{\sin 2x}{4} dx$$

$$= 2\pi\left(\frac{\pi}{2} - \frac{\pi^2}{8}\right) - 2\pi\left(-\cos x - \frac{x^2}{4} + \frac{\cos 2x}{8}\right)\Bigg|_0^{\frac{\pi}{2}} = \pi^2 - \frac{\pi^3}{4} - 2\pi\left(1 - \frac{\pi^2}{16} - \frac{1}{4}\right)$$

$$= -\frac{\pi^3}{8} + \pi^2 - \frac{3\pi}{2} = -\frac{\pi}{8}(12 - 8\pi + \pi^2)$$

4. Find the surface area of the volume formed by revolving $y = 1 - x^2$ for $0 \le x \le 1$, about the *y*–axis.
SOLN: $2\pi \int rds = 2\pi \int_0^1 x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 x\sqrt{1 + 4x^2}dx = \frac{\pi}{4}\int_1^5 \sqrt{u}du = \frac{\pi}{6}(5\sqrt{5} - 1)$

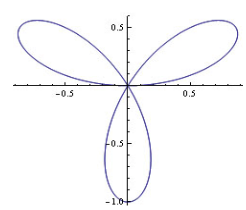
5. Set up an integral to compute the total fluid force on one side of the vertical plate submerged in water as shown at right. You don't need to evaluate the integral.
Hint: recall the element of force is
= force density * depth * element of surface area



SOLN: Introduce a coordinate system with origin at the tip top of the triangle and pointing straight down. Then the cross-sectional length will increase with *x* according to $l(x) = \frac{4x}{5}$, giving an element of surface area at a nearly constant depth *x* of $dA = \frac{4x}{5}dx$. Thus the total force = $9810 \int_0^5 \frac{4}{5}x^2 dx = 2616x^3|_0^5 = 2616(125) = 327000N$.

6. Find the arc length of the curve given by the parametric equations $x = 1 + t^2,\ y = 2 + t^3$ between the points (2, 3) and (5, 10) in the *xy*-plane.
SOLN: $L = \int ds = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 \sqrt{4t^2 + 9t^4}dt = \int_1^2 t\sqrt{4 + 9t^2}dt = \frac{1}{18}\int_{13}^{40} \sqrt{u}du = \frac{1}{27}u^{\frac{3}{2}}\Big|_{13}^{40} = \frac{40\sqrt{40} - 13\sqrt{13}}{27} = \frac{80\sqrt{10} - 13\sqrt{13}}{27}$

7. Find the area enclosed by the curve $r = \sin(3\theta)$.
The graph of this polar function is shown at right.



Hint: Recall that the form of a polar integral is $\int_a^b \frac{r^2}{2} d\theta$
SOLN: $\int_0^{\pi} \frac{\sin^2 3\theta}{2} d\theta = \frac{1}{4}\int_0^{\pi} 1 - \cos 6\theta\, d\theta = \frac{1}{4}\theta - \frac{\sin 6\theta}{6}\Big|_0^{\pi} = \frac{\pi}{4}$

8. Expand $f(x) = \sqrt[3]{1 + x^2}$ as a binomial series to obtain a 6th degree approximating polynomial.
SOLN: $\sqrt[3]{1 + x^2} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^{2n} \approx 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{5}{81}x^6$

9. Find the radius of convergence and interval of convergence of the power series, $f(x) = \sum_{n=0}^{\infty} \frac{(2x-3)^n}{n!}$.
SOLN: The series converges if $\lim_{n\to\infty} \left|\frac{(2x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2x-3)^n}\right| = \lim_{n\to\infty} \frac{2x-3}{n+1} = 0$ for all *x*. Thus the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$

10. Simplify the fifth degree Taylor polynomial for $f(x) = \sin 2x$ expanded about $a = \frac{\pi}{8}$.
$\sum_{n=0}^{5} \frac{f^{(n)}\left(\frac{\pi}{8}\right)}{n!}\left(x - \frac{\pi}{8}\right)^n = \frac{\sqrt{2}}{2} + \sqrt{2}\left(x - \frac{\pi}{8}\right) - \sqrt{2}\left(x - \frac{\pi}{8}\right)^2 - \frac{2\sqrt{2}}{3}\left(x - \frac{\pi}{8}\right)^3 + \frac{\sqrt{2}}{6}\left(x - \frac{\pi}{8}\right)^4 + \frac{\sqrt{2}}{15}\left(x - \frac{\pi}{8}\right)^5$