Math 1B **Name (Print):** ____________________

Chapters 8 and 10 Test

Write all responses on separate paper. Show your work for credit. You may consult with fellow students, but don't copy their work.

1. A thin sheet of metal is shaped like the region in the first quadrant between $y = \sin x$ and $y = \dfrac{x}{2}$.
   (a) Graph the region.
   (b) Find the area of the region.
   (c) Find the x-coordinate of the center of mass of the region.
   (d) Find the y-coordinate of the center of mass of the region.

2. An art student was given a circular metal disk two feet in diameter and told to drill a small hole in it so that when the disk is cut in half and the piece with the hole is placed atop a spike stuck in the hole, it will balance. Not knowing about integrals, the artist drilled a hole at a point half-way between the center and the edge.
   (a) Where *should* the student have drilled the hole?
   (b) Now that she has made the mistake, she decides that rather than drill a second hole, she will cut the piece with the hole in it in such a way that it will balance on the spike at the point of the hole. Explain clearly how the disk should be cut so that our artist friend can understand.
   (c) What is the area of the piece of the metal disk that balances at the point where the hole $\left(\dfrac{1}{2}, 0\right)$ was drilled?

3. In each of the following, find the length of the given arc and use Pappus' theorem to find the volume of the solid obtained by rotating the given region, $\mathscr{R}$
   (a) Arc: $y = \sin x$ from $x = 0$ to $x = \pi$.
   $\mathscr{R}$: One arch of the sine curve above the x–axis.
   (b) Arc: $y = x^2$ from $x = 0$ to $x = 1$.
   $\mathscr{R}$: The region between $yu = x^2$ and $y = \sin x$ in the first quadrant.
   (c) Arc: $y = \sqrt{1 - \dfrac{x^2}{9}}$ from $x = 0$ to $x = 3$ (The integral is a special case of an *elliptic integral.*
   $\mathscr{R}$: The region in the first quadrant that lies beneath $y == \sqrt{1 - \dfrac{x^2}{9}}$ and outside the unit circle.

4. In this problem we will examine the length of the arc of the curve $y = x^n$ on the interval $[0, 1]$ for different values of n.
   (a) Approximate the length of the arc of the curve $y = x^n$ on the interval $[0, 1]$ fpr $n = 1, 10, 20,$ and 100.
   (b) For the case $n = 1$ explain how you can get the answer very quickly by just looking at the graph.
   (c) Discuss any pattern or trend you see in the calculations in Part (a).
   (d) Plot the graphs of the four curves in Part 1 and use them to help explain what is happening to the arc lengths as n gets larger.
   (e) Based on all the above, find $\displaystyle\lim_{n\to\infty} \int_0^1 \sqrt{1 + (n+1)^2 x^{2n}}\, dx$
   (f) Repeat Parts (a) through (d) using the curve $y = \sqrt{1 - x^n}$ on the interval $[-1, 1]$.

5. We will explore what happens to the ratio of arc length to area on $[0,1]$ as $a \to \infty$ for four curves that depend on the parameter a. For each of the four functions that follow,
(i) Plot the graph of the function for $a = 1$.
(ii) Find the area bounded by the function and the x–axis on $[0,1]$.
(iii) With pencil and paper, write down the integral formulas for the arc length on $[0,1]$ and the area under the curve on $[0,1]$. Use these to find an integral formula for the limit of the ratio of arc length to area as $a \to \infty$.
(iv) Using your work in Part (iii), find the limit as $a \to \infty$ of the ratio of arc length to area on $[0,1]$
(v) By looking at the geometry of the graph, can you find a way to predict the limit in Part (iv) without doing the calculations?
   (a) $a(x - x^2)$
   (b) $a\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right)$
   (c) $a\sin(\pi x)$
   (d) a time a semicircle of radius 1.

6. Consider a flat metal plate to be placed vertically under water with its top 2 meters below the surface of the water. Determine a shape for the plate so that if the plate is divided into any number of horizontal strips of equal height, the hydrostatic force on each strip is the same.

7. Find the centroid of the region enclosed by the ellipse $x^2 + (x + y + 1)^2 = 1$. Note that it's tilted...

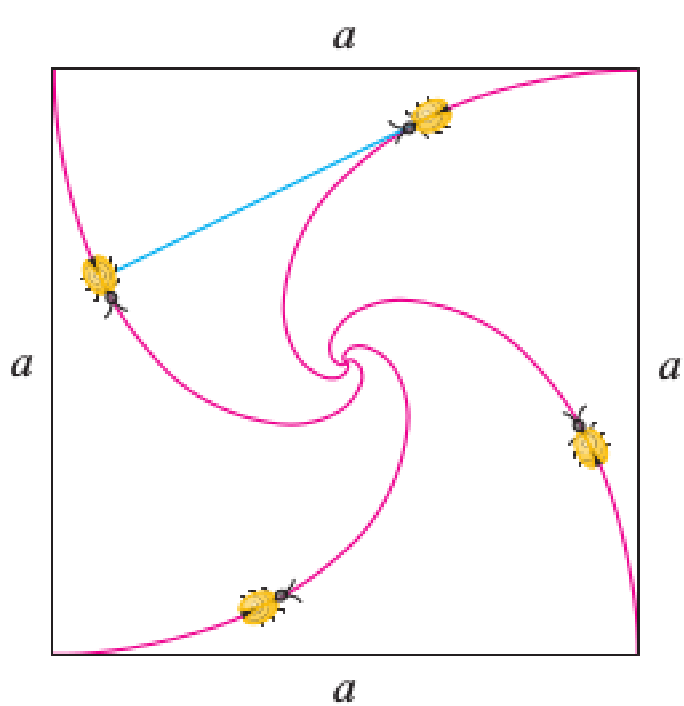
8. Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta), a \le \theta \le b$ (where f' is continuous and $0 \le a < b \le \pi$), about the line $\theta = \pi/2$. Apply this to $r = \cos(2\theta)$.

9. Find the arclength and area enclosed by

$$x = 2a\cos t - a\cos(2t)$$
$$y = 2a\sin t - a\sin(2t)$$

for $a = 1$ and $a = 2$

10. Four bugs are placed at the four corners of a square with side length a . The bugs crawl counter-clockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.



   (a) Find the polar equation of a bugs path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bugs path.)

   (b) Find the distance traveled by a bug by the time it meets the other bugs at the center.