Math 1B **Name (Print):** ____________________

Final Exam

Write all responses on separate paper. Show your work in detail for credit.

There are 8 problems from which you will choose 7. Indicate here which problem you'd like to omit: ☐

1. The graph below plots the velocity of a bus decelerating to a stop over a period of twenty seconds. Points from the graph are tabulated here:

t (sec)	0	2	4	6	8	10	12	14	16	18	20
v (m/sec)	10	9.9	9.7	8.9	6.7	3.3	1.1	0.3	0.1	0.0	0.0

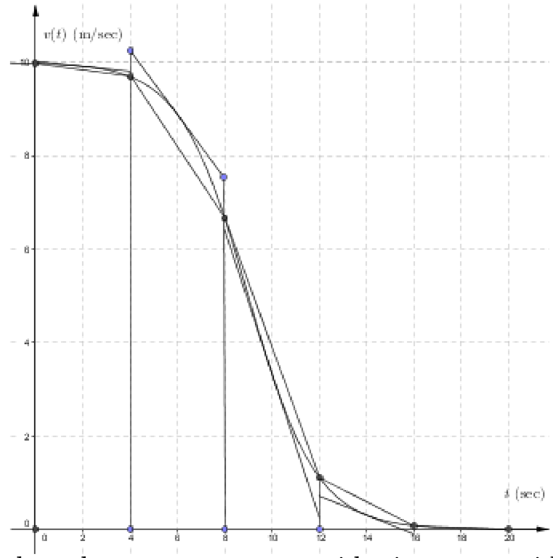
Approximate the distance traveled by the bus during this time with

(a) 5 subintervals of equal lengths and right endpoints as sample points.
SOLN: $4(9.7 + 6.7 + 1.1 + 0.1) = 70.4$ meters

(b) 5 trapezoids of equal widths.
SOLN: $4(5 + 9.7 + 6.7 + 1.1 + 0.1 + 0) = 90.4$

(c) 5 subintervals of equal lengths and midpoints as sample points.
SOLN: $4(9.9 + 8.9 + 3.3 + 0.3 + 0) = 89.6$

(d) Explain the relationship between the trapezoidal sum and the midpoint sum in terms of the concavity of the curve.



SOLN: As the diagram shows, the curve changes concavity so that the more accurate midpoint trapezoids are underestimates in the last three (somewhat smaller) areas than in the first two, and trapezoids the opposite. We know the Simpson estimate is $(2M + T)/3 \approx 89.9$, but the concavity change makes it harder to definitively claim that this in-between value is super-accurate, but it is the best estimate of all.

2. The Fresnel function $C(x) = \int_0^x \cos\left(\frac{1}{2}\pi t^2\right)\, dt$ is used to model the diffraction of light waves.

(a) Over what intervals is C increasing?
SOLN: C is increasing where $C'(x) = \cos\left(\frac{1}{2}\pi x^2\right) > 0 \Leftrightarrow \frac{(4k-1)\pi}{2} < \frac{1}{2}\pi x^2 < \frac{(4k+1)\pi}{2} \Leftrightarrow 4k - 1 < x^2 < 4k + 1$
Which means $(-1, 1)$ together with the intervals $\sqrt{4k-1} < x < \sqrt{4k+1}$ and $-\sqrt{4k+1} < x < -\sqrt{4k-1}$ for $k = 1, 2, 3, \ldots$.

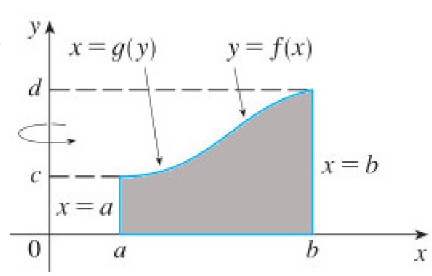
(b) Where are the inflection points of C?
SOLN: Look where the second derivative, $-\pi x \sin\left(\frac{1}{2}\pi x^2\right)$ changes sign. This is where $x = \pm\sqrt{2k}$ where $k \in \mathbb{Z}$.

3. In the diagram below, the region bounded by $x = 1, x = b, y = 0$ and $y = f(x) \Leftrightarrow x = g(y)$ is revolved around the y-axis to generate the volume, V.

(a) Use the diagram to explain that $V = \pi b^2 d - \pi a^2 c - \int_c^d \pi\left[g(y)\right]^2\, dy$

SOLN: $\pi b^2 d - \pi a^2 c$ is the volume of a cylinder with height d and radius b with the cylinder of height c and radius a removed from its lower center.

If we further remove the volume of revolution, $\int_c^d \pi\left[g(y)\right]^2\, dy$ then what remains is the volume generated by rotating the shaded region about the y-axis.



(b) Use $f(x) = 1 + (x-1)^2$ with $a = 1$ and $b = 2$ to compute the corresponding volume.

SOLN: Here, $g(y) = 1 + \sqrt{y-1}$ and we compute $V = \pi 2^2(2) - \pi 1^2(1) - \int_1^2 \pi \left[1+\sqrt{y-1}\right]^2 dy$
$= 7\pi - \pi \int_0^1 (1+\sqrt{u})^2\, du = 7\pi - \pi \left(u - \frac{4}{3}u^{3/2} + \frac{1}{2}u^2\right)_0^1 = 7\pi - \frac{\pi}{6} = \frac{41\pi}{6}$

4. Use integration by parts and substitution methods to evaluate the integral.

(a) $\int_0^{\pi/4} x \sin 2x\, dx$

$u = x$	$dv = \sin 2x\, dx$
$du = dx$	$v = -\frac{1}{2}\cos 2x$

$\int_0^{\pi/4} x\sin 2x\, dx = -\frac{x}{2}\cos 2x\Big|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\cos 2x\, dx = \frac{1}{4}$

(b) $\int_0^{\pi/4} x^2(\sin 2x)\, dx$

du/dx		$\int v dx$
x^2	$\searrow$	$\sin 2x\, dx$
$-2x$	$\searrow$	$-\frac{1}{2}\cos 2x\, dx$
2	$\searrow$	$-\frac{1}{4}\sin 2x$
0		$\frac{1}{8}\sin 2x$

$\frac{-x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}$

5. If $f(t)$ is continuous for $t \geq 0$, the Laplace transform of f is the function F defined by $F(s) = \int_0^\infty f(t)e^{-st}\, dt$.

Find the Laplace transforms of the following functions.

(a) $f(t) = 1$ SOLN: $F(s) = \int_0^\infty e^{-st}\, dt = \lim_{b\to\infty} \frac{-e^{-st}}{s}\Big|_0^b = \frac{1}{s}$

(b) $f(t) = t$ SOLN: $F(s) = \int_0^\infty te^{-st}\, dt =$

$u = t$	$dv = e^{-st}dt$
$du = dt$	$v = -\frac{e^{-st}}{s}$

$\lim_{b\to\infty} \frac{-te^{-st}}{s}\Big|_0^b + \int_0^b \frac{e^{-st}}{s}\, dt = \frac{1}{s^2}$

6. Consider the curve $y = \frac{x^4}{16} + \frac{1}{2x^2}$

(a) Find the length of the curve for $1 \leq x \leq 2$.
SOLN: $L = \int_1^2 \sqrt{1 + \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}}\, dx = \int_1^2 \sqrt{\frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6}}\, dx$
$= \int_1^2 \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2}\, dx = \frac{x^4}{16} - \frac{1}{2x^2}\Big|_1^2 = \frac{21}{16}$

(b) Find the area of the surface obtained by rotating the curve in part (a) about the y-axis.
SOLN: $A = 2\pi \int r\, ds = 2\pi \int_1^2 x\left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx = 2\pi\left(\frac{x^5}{20} - \frac{1}{x}\right)_1^2 = \frac{41\pi}{10}$

7. Use a Maclaurin series to approximate each to the nearest billionth (10^{-9}).

(a) Since the terms in the series for e^x are positive and decreasing very rapidly, we can approximate by looking at the first neglected term: $e^{0.1} \approx 1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{1}{240000} + \frac{1}{12,000,000} + \frac{1}{720,000,000} \approx 1.10517091806$ while $e^{0.1} \approx 1.10517091808$

(b) $\sqrt{10}$ Recall that $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$
SOLN: $\sqrt{10} = \sqrt{9+1} = 3\sqrt{1+1/9} \approx$
$3\left(1 + \frac{1}{2}\frac{1}{9} - \frac{1}{8}\frac{1}{81} + \frac{1}{16}\frac{1}{729} - \frac{1}{128}\frac{1}{6561} + \frac{7}{256}\frac{1}{59049} - \frac{21}{1024}\frac{1}{531441} + \frac{33}{2048}\frac{1}{4782969} - \frac{429}{32768}\frac{1}{43046721}\right) \approx 3.16227766009$
Compare with $\sqrt{10} \approx 3.16227766017$

8. Use series to approximate $\int_0^{0.5} x^2 e^{-x^2}\, dx$ to the nearest thousandth.

SOLN: $\int_0^{0.5} x^2 e^{-x^2}\, dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{x^2 \cdot (-x^2)^n}{n!}\, dx = \sum_{n=0}^{\infty} \int_0^{0.5} \frac{(-1)^n x^{2n+2}}{n!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)n!}\Big|_0^{0.5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3}(2n+3)n!}$

$\approx \frac{1}{24} - \frac{1}{160} + \frac{1}{1792} \approx 0.036$