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92. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that $\displaystyle\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

SOLN: If $u = \pi - x$ then $x = u - \pi, du = -dx$ and $\sin(x) = \sin(\pi - u) = \sin u$ (where this last equality follows from the supplementary identity: $\sin(\pi - \theta) = \sin\theta$.

Now $0 \leq x \leq \pi \Leftrightarrow 0 \leq \pi - u < \pi \Leftrightarrow pi \geq u \geq 0$ so $\int_0^{\pi} xf(\sin x)dx = \displaystyle\int_{\pi}^{0} (\pi - u)f(\sin(\pi - u)(-du)$. Reversing bound negates the $-du$ so $\displaystyle\int_{\pi}^{0} (\pi - u)f(\sin(\pi - u)(-du) = \int_0^{\pi} (\pi - u)f(\sin(\pi - u)(du)$ and applying supplemental identity gives us $\displaystyle\int_0^{\pi} (\pi - u)f(\sin(\pi - u)(du) = \int_0^{\pi} (\pi - u)f(\sin(u))du$

Now the integral of a difference is the difference of integrals:

$\displaystyle\int_0^{\pi} (\pi - u)f(\sin(u))(du) = \pi\int_0^{\pi} f(\sin(u))du - \int_0^{\pi} uf(\sin(u))du$

Now observe that the value of a definite integral is independent of the name of the variable of integration, so we can rename $u = x$ and get $\displaystyle\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi} f(\sin(x))dx - \int_0^{\pi} xf(\sin(x))dx$

If we name the original integral I we see that this equation has the form

$I = \pi\displaystyle\int_0^{\pi} f(\sin(x))dx - I$. Solving for I gives the desired result.