Math 1B Chapter 11 Test Solutions-Spring 2017

1. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

   (a) $\sum_{n=1}^{\infty} \ln \dfrac{n}{2n+1}$

   ANS: This fails the nth term test since $\lim_{n\to\infty} \ln \dfrac{n}{2n+1} = \ln \left(\lim_{n\to\infty} \dfrac{1}{2+1/n} \right) = -\ln 2 \neq 0 \Rightarrow$ divergent.

   (b) $\sum_{n=1}^{\infty} \dfrac{(-2)^n n}{9^n}$

   ANS: This is an alternating series where $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{4.5^n}$ =(L'Hospital's rule applies since both numerator and denominator go to ∞)= $\lim_{n\to\infty} \dfrac{\ln 4.5}{4.5^n} = 0$, so it converges.

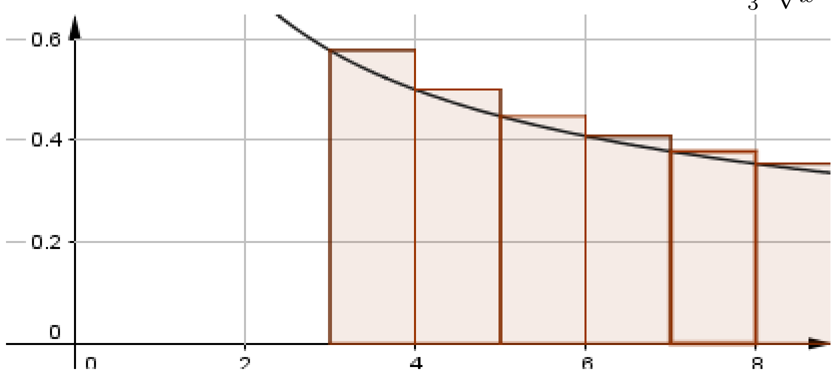
2. (14 points) Find the value of c if $\sum_{n=0}^{\infty} \left(\dfrac{c}{10}\right)^n = 100$.

   ANS: This has the form of a geometric series, it converges iff $\left|\dfrac{c}{10}\right| < 1$. If so, it converges to
   $\dfrac{1}{1-\frac{c}{10}} = \dfrac{10}{10-c} = 100 \Leftrightarrow 10 - c = \dfrac{1}{10} \Leftrightarrow \boxed{c = \dfrac{99}{10}}$

3. (18 points) Consider the series $S = \sum_{n=3}^{\infty} \dfrac{1}{\sqrt{n}}$.

   (a) Sketch a diagram (you can use the starter plot below, or make your own) to justify the inequality, $S > \int_3^{\infty} \dfrac{dx}{\sqrt{x}}$

   (b) Use the integral test to prove the series is divergent.
   As the diagram demonstrates, the area under the curve over $[3, \infty]$ is less than the area under the function $f(x) = \dfrac{1}{\sqrt{n}}$, which is infinite by the p-test $p \leq 1$ or by the integral test:
   $\int_3^{\infty} x^{-1/2}\, dx = \lim_{b\to\infty} 2x^{1/2}\Big|_3^b = \infty$



   (c) What is the smallest value of N for which you can be sure, based on the integral test, that $S_N = \sum_{n=3}^{N} \dfrac{1}{\sqrt{n}} > 100$

   ANS: $S_N > \int_3^{N+1} \dfrac{dx}{\sqrt{x}} = 2x^{1/2}\Big|_3^{N+1} = 2\sqrt{N+1} - 2\sqrt{3} > 100$
   $\Leftrightarrow N + 1 > (50 + \sqrt{3})^2 \Leftrightarrow N > 2500 + 100\sqrt{3} + 3 - 1 \approx 2502 + 100(1.732) = 2675$
   Indeed, you can write a little Python script to check this:

```python
import math

sum = 0
for x in range(3,2675):
    sum += 1/math.sqrt(x)
sum
```

   which returns a number a little bit bigger than 100: 100.26367537183538
   Of course, a smaller number will do: I think 2662 is good enough.

4. (16 points) (a) Use the limit comparison test to prove that $\sum_{n=7}^{\infty} \dfrac{n-2}{n(n-5)}$ is divergent.

   ANS: Do limit comparison with the harmonic series: $\lim_{n\to\infty} \dfrac{\frac{n-2}{n(n-5)}}{1/n} = \lim_{n\to\infty} \dfrac{n^2-2n}{n^2-5n} = 1$, and since the harmonic series is divergent, so is this series.

(b) Show that $\sum_{n=7}^{\infty} \frac{(-1)^{n+1}(n-2)}{n(n-5)}$ is convergent. Find the smallest N so that $R_N = S - S_N < 0.001$

ANS: This is an alternativing series and the nth term is steadily decreasing, approaching 0 in the limit as $n \to \infty$, so it's convergent. The error in approximation is less than the first neglected term, so we want to find N so that $\frac{N-1}{(N+1)(N-4)} < 0.001 \Leftrightarrow 1000N - 1000 < N^2 - 3N - 4 \Leftrightarrow N^2 - 1003N + 996 > 0$

Completing the square, $\left(N - \frac{1003}{2}\right)^2 > \left(\frac{1003}{2}\right)^2 - 996 \Leftrightarrow N > 501.5 + \sqrt{\frac{1006009}{4} - 996} \approx 501.5 + 500 \approx 1000$

5. (24 points) Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (definition of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for e^x, $\sin(x)$, $\tan^{-1} x$, and $\ln(1+x)$.

(a) $f(x) = \frac{x}{1+x^3}$.

ANS: $f(x) = x \cdot \frac{1}{1-(-x^3)} = x \cdot \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n+1}$. By the ratio test, $\lim_{n\to\infty} |\frac{x^{3n+4}}{x^{3n+1}}| = |x^3| < 1 \Leftrightarrow -1 < x < 1$ and thus the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

(b) $f(x) = \ln(1-x^2)$.

$f'(x) = \frac{-2x}{1-x^2} = -2x \cdot \sum_{n=0}^{\infty} (x^2)^n = -2 \cdot \sum_{n=0}^{\infty} x^{2n+1}$. Integrating to recover $f(x) = -2 \cdot \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2}$. By the ratio test, $\lim_{n\to\infty} \frac{x^{2n+4}(2n+2)}{x^{2n+2}(2n+4)} = x^2 < 1 \Leftrightarrow -1 < x < 1$

(c) $f(x) = \frac{x}{\sqrt{4+x^2}} = \frac{x}{2}\left(1 + \frac{x^2}{4}\right)^{-1/2} = \frac{x}{2} \cdot \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x^{2n+1}}{2 \cdot 4^n}\right)$. The ratio test gives us $\lim_{n\to\infty} \left| \frac{\binom{-1/2}{n+1} x^{2n+3} \cdot 4^n}{\binom{-1/2}{n} x^{2n+1} \cdot 4^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2(-\frac{1}{2} - n)}{4(n+1)} \right| = \frac{x^2}{4} < 1 \Leftrightarrow -2 \le x \le 2$. For $x = \pm 2$, the series is alternating and satisfies the alternating series test: $f(\pm 2) = \sqrt{2}/2 = \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{2^{2n+1}}{2^{2n+1}}\right) = \sum_{n=0}^{\infty} \binom{-1/2}{n}$.

Shall we check that?

```
import scipy.special
sum = 0
for n in range(10000000):
    sum += scipy.special.binom(-0.5, n)
sum
```

produces (after a pause for "thought") 0.7070175800236228 So it takes ten million terms to get the first three digits, but it seems to approaching $\sqrt{2}/2 \approx$`print(math.sqrt(2)/2)=0.7071067811865476`