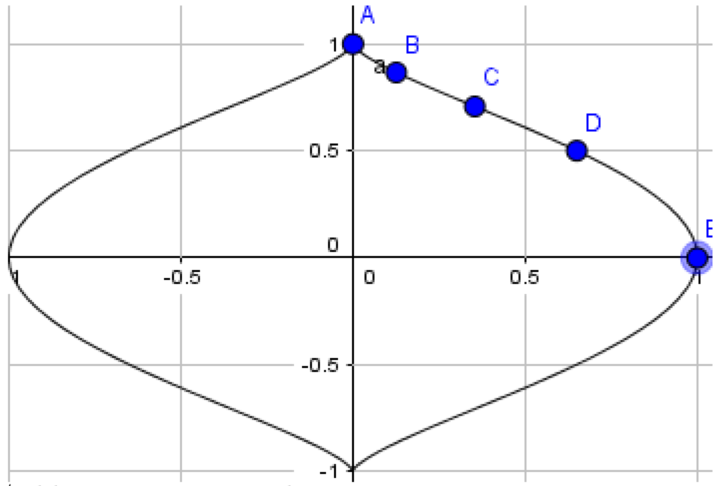
1. (16 points) Consider the curve defined by the parametric equations

$$x = \sin^3(t)$$
$$y = \cos(t)$$

(a) Complete the table of values and extend it as needed to sketch a complete graph for the the curve in the xy-plane.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x	0	$\frac{1}{8}$	$\frac{\sqrt{2}}{4}$	$\frac{3\sqrt{3}}{8}$	1
y	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0



Solution: The table is as completed and even/odd symmetry is used to generate symmetric points in the other quadrants for the plot above. Oniony.

(b) Find the coordinates of the points of inflection.

Solution: The inflection points are where the second derivative, $\frac{d^2y}{dx^2}$ changes sign. Start by computing $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin(t)}{3\cos(t)\sin^2(t)} = -\frac{1}{3\sin(t)\cos(t)} = -\frac{2}{3\sin(2t)}$. Then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx}\frac{d}{dt}\left(-\frac{2}{3\sin(2t)}\right) = \frac{1}{3\cos(t)\sin^2(t)}\left(-\frac{4\cos(2t)}{3\sin^2(2t)}\right) = \frac{-\cos(2t)}{3\sin^4 t\cos^3 t}$ The definition for *inflection point* is

A point P on a curve is called an ***inflection point*** if it is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Certainly, the points where $t = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$ meet that description. The other points where the concavity changes are $t = \frac{(2k+1)\pi}{4}, k \in \mathbb{Z}$ that is, $\left(\pm\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}\right)$ These points include the point labeled "C" in the diagram above, and its reflection across the x-axis and y-axis.

2. (16 points) Consider the parametric equations,

$$x(t) = \int_0^t \cos(u^2)\,du, \qquad y(t) = \int_0^t \sin(u^2)\,du$$

(a) Find the the arclength of this curve over the interval from $t \in [0, s]$.

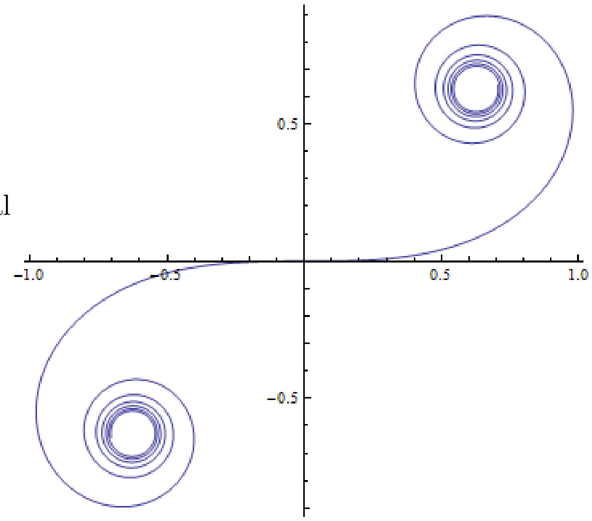
Solution: Arclength is $L = \int ds = \int_0^s \sqrt{(x'(t))^2 + (y'(t))^2}dt$ These derivatives are easily computed using the fundamental theorem of calculus. For example, $x'(t) = \frac{d}{dt}\int_0^t \cos(u^2)\,du = \cos^2(t)$ so

$L = \int_0^s \sqrt{\cos^2(t) + \sin^2(t)}dt = s.$

(b) For what values of t is the line tangent to the curve vertical?

Solution: The tangent line is vertical if $x'(t) = \dfrac{d}{dt}\int_0^t \cos(u^2)\,du = \cos(t^2) = 0$, that is, where $\boxed{t = \pm\sqrt{\pm\dfrac{\pi}{2} + 2k\pi}}$

As a follow up, we can plot the curve in Mathematica like so:
x[t_]:=Integrate[Cos[u^2], {u, 0, t}]
y[t_]:=Integrate[Sin[u^2], {u, 0, t}]
ParametricPlot[{x[t],y[t]},{t,-2*Pi,2*Pi}]
This produces the curve below called the Cornu Spiral - a structure with deep meaning for the universe.



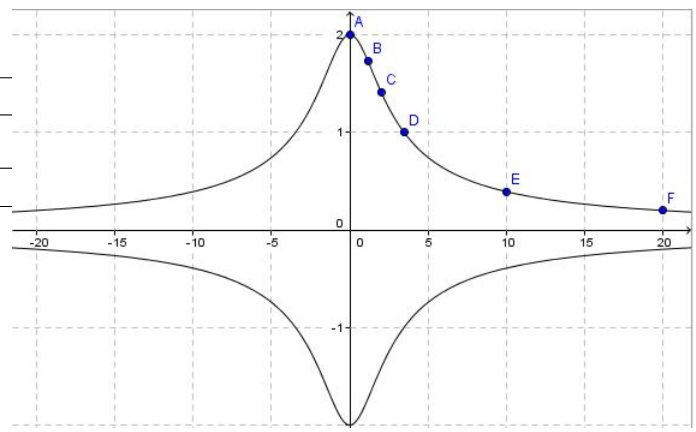
3. (16 points) Consider the parametric equations

$$x(\theta) = 2\tan\theta$$
$$y(\theta) = 2\cos\theta$$

(a) Make a table & sketch a graph for the curve.

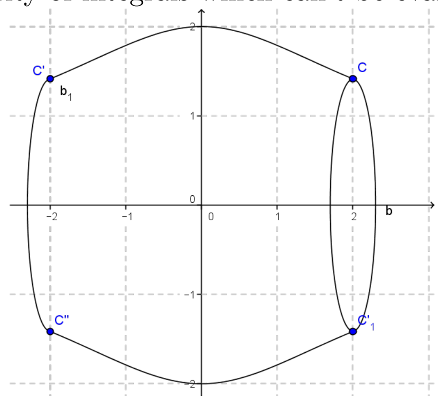
Solution:

label	A	B	C	D	E	F
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	arctan(5)	arctan(10)
x	0	$2\sqrt{3}/3$	2	$2\sqrt{3}$	10	20
y	2	$\sqrt{3}$	$\sqrt{2}$	1	$\frac{\sqrt{26}}{26}$	$\sqrt{101}101$



(b) Set up and simplify an integral for the surface area generated when the part of the curve $t \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ is revolved about the x-axis.

An infinitesimal piece of area for the surface of the volume of revolution is given by $dA = 2\pi r ds$ where $ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt = \sqrt{4\sec^4(t) + 4\sin^2(t)}dt$. The surface area is then $A = 8\pi \int_{-\pi/4}^{\pi/4} \sec\theta\sqrt{1 + \cos^4\theta\sin^2\theta}\,d\theta$
There is no obvious substitution to simplify this integral, but is there an unobvious one? Turning to the CAS world, we try Mathematica, which spits back $16\pi\int_0^{\frac{\pi}{4}}$ Sec[t]$\sqrt{1 + \text{Cos}[t]^4\text{Sin}[t]^2}\,dt$. That makes it look very much like this is one of the overwhelming majority of integrals which can't be evaluated by finding an

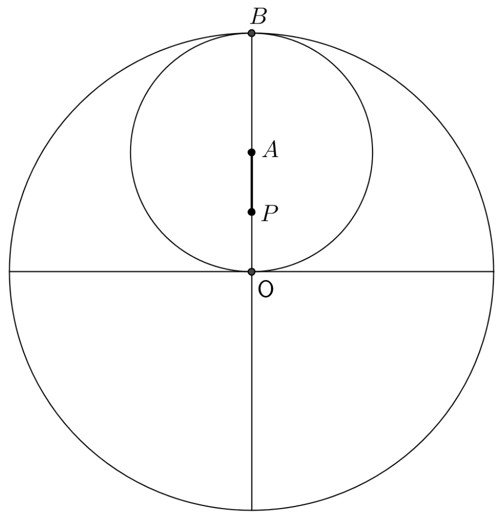


elementary antiderivative. Try the clamped method!

4. (20 points) In the diagram, a circle of radius $OA = 2$ rolls around the inside of a circle of radius $OB = 4$ with the half radius $AP = 1$ originally positioned so that P is on OB. Find a parameterization for the path that P traces out as the inner circle rolls (without slipping.) Start by choosing an appropriate parameter. Is that a familiar curve?

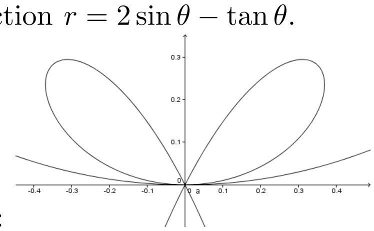
   ANS: The key here is to assert that the arclength swept out by rolling is the same on both circles, so if θ is the angle swept out by $\overrightarrow{OA}$ as the circle rolls and α is the angle swept out by $\overrightarrow{AP}$ then the arclengths are equal so $4\theta = 2\alpha \Rightarrow \alpha = 2\theta$. However, θ is subtracted from α because as the inner circle rolls, it both rotates around its center and rotates around the bigger circle's center; at the same time the inner circle's center is rotating around the bigger circle's center, so the angle from the vertical of the line $\overline{AP}$ is $2\theta - \theta = \theta$. You can think of the process of getting from the origin (0,0) to the point P as a two step process: first get from (0,0) to $A = (2\sin\theta, 2\cos\theta)$ and then get from A to P, which is the vector $\overrightarrow{AP} = (\sin\theta, -\cos\theta)$, thus the coordinates of P are $(3\sin\theta, \cos\theta)$. Since these coordinates satisfy the equation $\dfrac{x^2}{9} + y^2 = 1$ the equation represents an ellipse.

   see `http://geofhagopian.net/animations/ellipsogon.mp4`



   or `http://www.geogebratube.org/student/m114354`

5. (16 points) Find the area enclosed by one of the loops in the graph of the polar function $r = 2\sin\theta - \tan\theta$.



   Solution: Sketch a graph of the figure with the command in Geogebra produces loops:
   To find the limits of these loop, find the angles corresponding to $r = 0$: $r = 0 \Leftrightarrow 2\sin\theta - \tan\theta = \sin\theta(2 - \sec\theta) = 0$ iff $\sin\theta = 0$ or $\cos\theta = \dfrac{1}{2} \Leftrightarrow \theta = \pm\dfrac{\pi}{3}$. Evidently a loop is formed as $0 \le \theta \le \dfrac{\pi}{3}$ so the area in a loop is

$$\int_0^{\pi/3} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/3} (2\sin\theta - \tan\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/3} (4\sin^2\theta - 4\frac{\sin^2\theta}{\cos\theta} + \tan^2\theta)\,d\theta =$$

$$\int_0^{\pi/3} (1 - \cos(2\theta))d\theta - 2\int_0^{\pi/3} \frac{1 - \cos^2\theta}{\cos\theta}d\theta + \frac{1}{2}\int_0^{\pi/3} \sec^2\theta - 1\,d\theta =$$

$$\left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/3} - 2(\ln|\sec\theta + \tan\theta| - \sin\theta)\Big|_0^{\pi/3} + \frac{1}{2}(\tan\theta - \theta)\Big|_0^{\pi/3}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} - 2\ln(2+\sqrt{3}) + \sqrt{3} + \frac{1}{2}\sqrt{3} - \frac{\pi}{6} = \frac{\pi}{6} + \frac{5\sqrt{3}}{2} - 2\ln(2+\sqrt{3}) \approx 0.0547465$$

   Wow. You suppose there was a mistake in there? Entering Integrate[(2*Sin[t]-Tan[t])∧2,{t,0,Pi/3}]/2 into Mathematica, we get $\frac{1}{2}\left(\frac{5\sqrt{3}}{2} + \frac{\pi}{3} + 4\text{Log}\left[-1+\sqrt{3}\right] - 4\text{Log}\left[1+\sqrt{3}\right]\right)$ which approximates as `0.0547465`. So it looks good!

6. (16 points) Use Simpson's rule with $n = 6$ to approximate the arc length of $r = \cos(4\cos(\theta))$ on the interval $t \in \left[0, \dfrac{\pi}{2}\right]$.

Solution: In polar coordinates we approximate ds as the "hypotenuse" of a triangle with "legs" dr and $r\,d\theta$ (draw the picture, to be sure), $ds \approx \sqrt{(dr)^2 + (r\,d\theta)^2} = \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$.

Now $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\cos(4\cos(\theta)) = 4\sin\theta\sin(4\cos\theta)$

So arclength is $\int_0^{\pi/2} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{\pi/2} \sqrt{\cos^2(4\cos\theta) + 16\sin^2\theta\sin^2(4\cos\theta)}\,d\theta$

We're approximating the integral of $f(\theta) = \sqrt{\cos^2(4\cos\theta) + 16\sin^2\theta\sin^2(4\cos\theta)}$ on $[a, b] = [0, pi/2]$ with Simpson's method, $n = 6$. Thus we'll use $h = \dfrac{b-a}{6} = \dfrac{\pi}{12}$ giving arclength $\approx \dfrac{h}{3}(f(0) + 4f(h) + 2f(2h) + 4f(3h) + 2f(4h) + 4f(5h) + f(6h))$ This isn't terribly onerous to do by hand, but I prefer to use my nifty TI-85 program (from the last project.) Entering our function for `y1` and running the program with $A = 0$, $b = \pi/2$ and $N = 6$ produces the following:

```
S=
        2.87624924113
C=
         2.8397684574
        -2.18310206057
        -2.14662127684
                  Done
```

Using Mathematica, we can validate this as follows:

$f[x\_] := Sqrt[(Cos[4 * Cos[x]])^2 + 16 * (Sin[x])^2 * (Sin[4 * Cos[x]])^2]$

f[0]

-Cos[4]

h:=Pi/12

f[h]

$Sqrt[Cos[Sqrt[2](1 + Sqrt[3])]^2 + 2(-1 + Sqrt[3])^2 Sin[Sqrt[2](1 + Sqrt[3])]^2]$

N[h/3*(f[0]+4*f[h]+2*f[2*h]+4*f[3*h]+2*f[4*h]+4*f[5*h]+f[6*h])]

2.87625