1. Evaluate the integral. Make all the substitutions and infinitesimals explicit.

   (a) $\int_0^\pi x\sin 3x\,dx.$

   ANS: $\boxed{\begin{matrix} u = x & dv = \sin 3x\,dx \\ du = dx & v = -\frac{1}{3}\cos 3x \end{matrix}}$ gives $\int_0^\pi x\sin 3x\,dx = -\frac{x}{3}\cos 3x\Big|_0^\pi + \frac{1}{3}\int_0^\pi \cos 3x\,dx = \frac{\pi}{3} + \frac{1}{9}\sin 3x\Big|_0^\pi = \boxed{\frac{\pi}{3}}$

   (b) $\int_1^e t^2 \ln t\,dt$ *Hint:* Let $u = \ln t$

   ANS: $\boxed{\begin{matrix} u = \ln t & dv = t^2\,dt \\ du = \frac{dt}{t} & v = \frac{1}{3}t^3 \end{matrix}}$ gives $\int_1^e t^2\ln t\,dt = \frac{t^3\ln t}{3}\Big|_1^e - \frac{1}{3}\int_1^e t^2\,dt = \frac{e^3}{3} - \frac{1}{9}t^3\Big|_1^e = \boxed{\frac{2e^3+1}{9}}$

2. Derive the reduction formula $\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$. Start be rewriting the integral as $\int \cos^{n-1}(x)\cos x\,dx$, and then doing integration by parts with

$$\boxed{\begin{matrix} u = \cos^{n-1}x & dv = \cos x\,dx \\ du = -(n-1)\cos^{n-2}x\sin x\,dx & v = \sin x \end{matrix}}$$

   Show the rest of the derivation, being careful not to skip any steps.

   ANS: $I_n = \int \cos^{n-1}(x)\cos x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx =$

   $\cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x(1-\cos^2 x)\,dx = \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n$

   $\Leftrightarrow nI_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}\boxed{\Leftrightarrow I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-2}}$

3. Find the volume obtained by rotating the region bounded by $y = \cos^2 x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $y = 0$ @ x-axis.

   ANS: $V = \int_{-\pi/2}^{pi/2} \pi y^2\,dx = 2\pi\int_0^{\pi/2}\cos^4 x\,dx = \frac{\pi}{2}\cos^3 x\sin x\Big|_0^{\pi/2} + \frac{3\pi}{2}\int_0^{\pi/2}\cos^2 x\,dx = \frac{3\pi}{4}\cos x\sin x\Big|_0^{\pi/2} + \frac{3\pi}{4}\int_0^{\pi/2}dx = \boxed{\frac{3\pi^2}{8}}$

4. Evaluate each integral:

   (a) $\int_0^{\frac{1}{4}} \sqrt{1-4x^2}\,dx$

   ANS: Let $4x^2 = \sin^2\theta \Leftarrow 2x = \sin\theta \Rightarrow 2dx = \cos\theta d\theta$, then $\int_0^{\frac{1}{4}}\sqrt{1-4x^2}\,dx = \int_0^{\pi/6}\sqrt{1-\sin^2\theta}\frac{1}{2}\cos\theta d\theta$

   $= \frac{1}{2}\int_0^{\pi/6}\cos^2 x\,dx = \frac{1}{4}\cos x\sin x\Big|_0^{\pi/6} + \frac{1}{4}\cdot\frac{\pi}{6} = \frac{\sqrt{3}}{16} + \frac{\pi}{24}$

   (b) $\int_1^2 \frac{6x^2+5x+4}{x(x^2+x+1)}\,dx = \int_1^2 \frac{A}{x} + \frac{Bx+C}{x^2+x+1}\,dx$ where

   $A(x^2+x+1) + (Bx+C)x = (A+B)x^2 + (A+C)x + A = 6x^2+5x+4 \Rightarrow A = 4, B = 2, C = 1$

so $= \int_1^2 \frac{4}{x} + \frac{2x+1}{x^2+x+1}\,dx = 4\ln x + \ln(x^2+x+1)\Big|_1^2 = 4\ln 2 + \ln 7 - \ln 3 = \ln\left(\frac{112}{3}\right)$

(c) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(x-\frac{1}{2})^2}{(3+4x-4x^2)^{3/2}}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(x-\frac{1}{2})^2}{(-4(x-\frac{1}{2})^2+4)^{3/2}}\,dx.$

As a baby step, let $u = x - \frac{1}{2}$, so the integral becomes $\int_{-1}^{0} \frac{u^2}{8(1-u^2)^{3/2}}du$

We'd like to eliminate the root by employing the $1-\sin^2\theta = \cos^2\theta$ identity. Let $u = \sin\theta, du = \cos\theta d\theta$. Then the integral becomes $\int_{-\pi/2}^{0} \frac{\sin^2\theta}{8\cos^3\theta}\cos\theta d\theta = \frac{1}{8}\int_{-\pi/2}^{0} \tan^2\theta d\theta = \frac{1}{8}\int_{-\pi/2}^{0} (\sec^2\theta - 1)d\theta = \lim_{b\to -\pi/2^+} \frac{1}{8}(\tan\theta - \theta)\Big|_b^0$, which does not exist.

5. Evaluate the improper integral.

(a) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}} = \lim_{b\to\frac{1}{2}^-} \int_0^b \frac{dx}{\sqrt{1-4x^2}} = \lim_{b\to\frac{1}{2}^-} \frac{1}{2}\int_0^{\arcsin(2b)} d\theta = \boxed{\frac{\pi}{4}}$ where $x = \frac{1}{2}\sin\theta, dx = \frac{1}{2}\cos\theta d\theta$

(b) $\int_1^{\infty} \frac{e^{-1/x^2}}{x^3}\,dx$. Let $u = \frac{-1}{x^2}$

so $du = \frac{2}{x^3}dx$. Note that, as $x\to\infty, u\to 0^+$, so the integral becomes $\frac{1}{2}\int_{-1}^{0} e^u du = \frac{1}{2}e^u\Big|_{-1}^0 = \boxed{\frac{e-1}{2e}}$

6. Use comparison to determine whether or not the integral is convergent.

(a) $\int_0^1 \frac{\cos^2 x}{\sqrt{x}}dx$

We start with the intuition that this behaves like $x^{-1/2}$ on $[0,1]$, which is convergent. So we observe that $0 < \cos^2 x \le 1$ and thus $\int_0^1 \frac{\cos^2 x}{\sqrt{x}}dx \le \int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{b\to 0^+}\int_b^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \frac{1}{2}\sqrt{x}\Big|_b^1 = \frac{1}{2}$, which shows that the integral is convergent.

(b) $\int_0^1 \frac{\tan^2 x}{x\sqrt{x}}\,dx$

We know that $\lim_{x\to 0}\frac{\tan x}{x} = 1$, so $\lim_{x\to 0}\frac{\tan^2 x}{x^{3/2}} = \lim_{x\to 0}\frac{\sqrt{x}\tan^2 x}{x^2} = \lim_{x\to 0}\sqrt{x}\left(\lim_{x\to 0}\frac{\tan x}{x}\right)^2 = 0.$

Thus $0 \le \frac{\tan^2 x}{x^{3/2}} \le 3$ on $[0,1]$ and we conclude that $\int_0^1 \frac{\tan^2 x}{x\sqrt{x}}\,dx \le \int_0^1 3\,dx = 3$ and the integral is convergent.