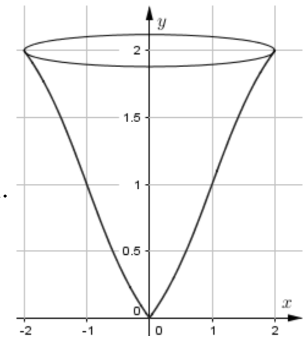
Math 1B Chapter 6-§7.1 Test Solutions

1. (20 points) The region in the first quadrant below $y = 2$ and above the curve $y = 1 + \frac{4}{\pi}\arctan(x-1)$ for $0 \le x \le 2$ generates the volume depicted at right when revolved about the $y-$axis.

   (a) Set up (but do not evaluate) an integral to compute the volume by the method of cylindrical shells.

   ANS: $V = \int dV = 2\pi \int_0^2 rh\,dx = 2\pi \int_0^2 x\left(2 - \left(1 + \frac{4}{\pi}\arctan(x-1)\right)\right) dx$

   $= 2\pi \int_0^2 x\left(1 - \frac{4}{\pi}\arctan(x-1)\right) dx == 2\pi \int_0^2 x\left(1 - \frac{4}{\pi}\arctan(x-1)\right) dx$

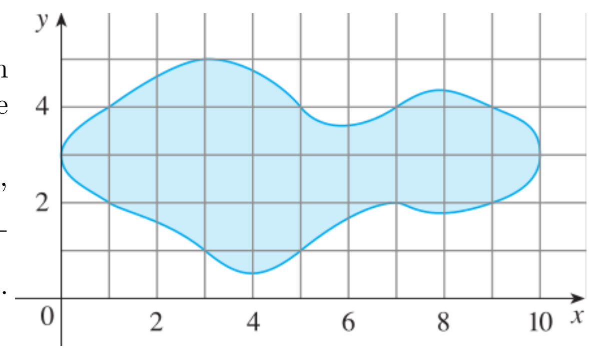


   (b) Set up (but do not evaluate) an integral to compute the volume by the disk method.

   ANS: Solve for x: $V = \int dV = \pi \int_0^2 r^2\,dy = \pi \int_0^2 \left(1 + \tan\left(\frac{\pi}{4}(y-1)\right)\right)^2 dy$

   (c) If the volume is filled with water, set up (but do not evaluate) and integral to compute the minimum work required to pump the water out over the top. Assume all units are MKS (meter, kilogram, second) and that the weight density of water is 9800 Newtons per cubic meter.

   ANS: An infinitesimal piece of volume at an equipotential in the gravitational field will be $dV = \pi\left((1 + \tan\left(\frac{\pi}{4}(y-1)\right)\right)^2$ and so an infinitesimal piece of weight will be $dF = 9800\pi\left((1 + \tan\left(\frac{\pi}{4}(y-1)\right)\right)^2 dy$. This weight will need to be lifted a distance $2 - y$ so we compute the work as $W = \int dW = 9800\pi \int_0^2 (2-y)\left(1 + \tan\left(\frac{\pi}{4}(y-1)\right)\right)^2 dy$



If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 5$ to estimate the volume of the solid.

2. ANS: We add up the volumes of a sequence of cylindrical shells, using midpts for the heights: $V \approx \sum_{i=1}^{5} h_i^*\pi(R_i^2 - r_i^2) = \pi(2(2^2 - 0^2) + 4(4^2 - 2^2) + 3(6^2 - 4^2) * 2(8^2 - 6^2) + 2(10^2 - 8^2)) = 244\pi$ cubic units.

3. (20 points) Evaluate the integral. Make all the substitutions and infinitesimals explicit.

   (a) $\int_0^{\pi/2} x^2 \sin 2x\,dx$. ANS: $\boxed{\begin{array}{ll} u = x^2 & dv = \sin 2x\,dx \\ du = 2x dx & v = -\frac{1}{2}\cos 2x \end{array}} \Rightarrow -\frac{1}{2}x^2 \cos 2x\Big|_0^{\pi/2} + \int_0^{\pi/2} x \cos 2x\,dx$

   $= \frac{\pi^2}{8} + \int_0^{\pi/2} x\cos 2x\,dx \boxed{\begin{array}{ll} u = x & dv = \cos 2x\,dx \\ du = dx & v = \frac{1}{2}\sin 2x \end{array}} = \frac{\pi^2}{8} + \frac{x}{2}\sin 2x\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \sin 2x\,dx = \frac{\pi^2}{8} + \frac{1}{4}\cos 2x\Big|_0^{\pi/2}$

   $\boxed{\frac{\pi^2}{8} - \frac{1}{2}}$

   (b) $\int_0^1 xe^{-2\pi x}\,dx$. ANS: $\boxed{\begin{array}{ll} u = x & dv = e^{-2\pi x}\,dx \\ du = dx & v = -\frac{1}{2\pi}e^{-2\pi x} \end{array}} \Rightarrow -\frac{1}{2\pi}\left[xe^{-2\pi x}\Big|_0^1 - \int_0^1 e^{-2\pi x}\,dx\right] = -\frac{e^{-2\pi}}{2\pi} - \frac{1}{4\pi^2}e^{-2\pi x}\Big|_0^1 =$

   $\boxed{= -\frac{1}{2\pi}\left[e^{-2\pi}\left(1 + \frac{1}{2\pi}\right) - \frac{1}{2\pi}\right] = \frac{1}{4\pi^2} - \frac{e^{-2\pi}}{2\pi} - \frac{e^{-2\pi}}{4\pi^2}}$

4. A bucket that weighs 4 Newtons and a rope of negligible weight are used to draw water from a well that is 100 meters deep. The bucket is filled with 36 Newtons of water and is pulled up at a rate of 0.5 meter / second, but water leaks out of a hole in the bucket at a rate of 0.2 Newtons / second. Assume that the bucket is filled at the bottom of the well and starts moving upward at time $t = 0$

   (a) What is the weight of the bucket (in Newtons) at time t seconds? (This is a decreasing linear function.)
   ANS: Actually, as Robert points out, this is a piecewise linear function, since it takes $36\texttt{N}/(0.2\texttt{N/s}) = 180\texttt{s}$ (3 minutes) for the bucket to drain, but $100\texttt{m}/(0.5\texttt{m/s}) = 200\texttt{s}$ (20 seconds longer) for the bucket to be raised to the top. Thus the weight of the bucket is given by the function

$$F(t) = \begin{cases} 40 - 0.2t & : 0 \le t \le 180 \\ 4 & : 180 < t \le 200 \end{cases}$$

   (b) What is the infinitesimal work done in the infinitesimal time interval dt ?
   ANS: $dW = F dx = F \frac{dx}{dt} dt = F \cdot 0.5dt =$

$$dW = \begin{cases} (40 - 0.2t)0.5dt & : 0 \le t \le 180 \\ 4 \cdot 0.5dt & : 180 < t \le 200 \end{cases}$$

   (c) Find the work done in pulling the bucket to the top of the well.
   Integrating over time, $W = \int dW = \int_0^{180} \left(20 - \frac{t}{10}\right) dt + \int_{180}^{200} 2\, dt = \left(20t - \frac{t^2}{20}\right)\Big|_0^{180} + 80 = 3600 - 1620 + 80$
   $\boxed{= 2100 \text{ Joules}}$

5. (20 points) (a) Derive the reduction formula $\boxed{I_n = \int \tan^n(x)\, dx = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}}$. Assume $n \ge 2$.
   Hint: use the identity $\tan^2 x = \sec^2 x - 1$ to break the integral into two pieces.
   ANS: Nice hint! Thus $I_n = \int \tan^n(x)\, dx = \int \tan^{n-2}(x) \tan^2 x\, dx = \int \tan^{n-2}(x)(\sec^2 x - 1)\, dx$
   $= \int \tan^{n-2}(x) \sec^2 x\, dx - \int \tan^{n-2}(x)\, dx = \int \tan^{n-2}(x) \sec^2 x\, dx - I_{n-2}$
   Substitute $u = \tan x, du = \sec^2 x\, dx$ so this becomes $I_n = \int u^{n-2} du - I_{n-2} = \frac{1}{n-1} u^{n-1} - I_{n-2}$
   $\boxed{= \frac{1}{n-1} \tan^{n-1} - I_{n-2}}$

   (b) Use the reduction formula to evaluate $\int_0^{\pi/4} \tan^3 x\, dx$

   ANS: $\int_0^{\pi/4} \tan^3 x\, dx = \frac{1}{2} \tan^2 x \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx = \frac{1}{2} - \ln(\sec x)\Big|_0^{\pi/4} \boxed{= \frac{1}{2} - \ln(\sqrt{2})}$