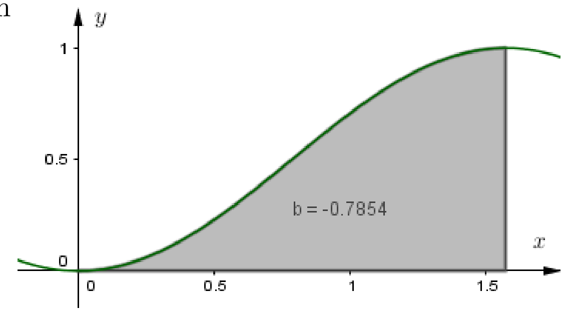
Math 1B Chapter 5&§6.1 Test Solutions

1. (15 points) Recall the definition for the **area** A of the region under a graph of a continuous function as the limit of the sum of areas of approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

We will use this to set up the computation for the area under $f(x) = \sin^2 x$ above $\left[0, \frac{\pi}{2}\right]$.

(a) Draw a diagram illustrating this region. SOLN at right:



$$\int_0^{\pi/2} \frac{1-\cos(2x)}{2}dx = \frac{x}{2} - \frac{\sin(2x)}{4}\Big|_0^{\pi/2} = \frac{\pi}{4}$$

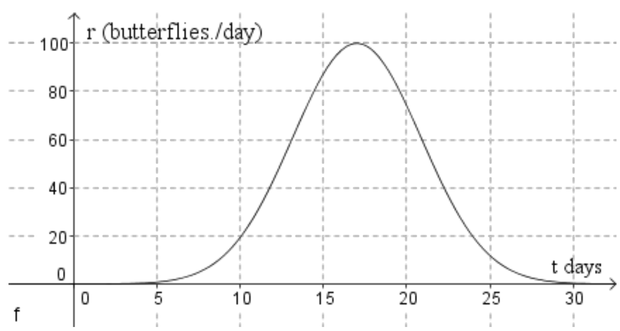
(b) Express Δx in terms of the number of intervals, n, in the partition of $\left[0, \frac{\pi}{2}\right]$. SOLN: $\boxed{\Delta x = \frac{\pi}{2n}}$

(c) Express the ith point in the partition, x_i, in terms of i and n.
SOLN: $\boxed{x_i = \frac{i\pi}{2n}}$

(d) Express the area as a limit involving only the variables n and i.
SOLN: $\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \sin^2\left(\frac{i\pi}{2n}\right)\frac{\pi}{2n}}$

(e) Use the identity $\sin^2 x = \frac{1-\cos(2x)}{2}$ and the Fundamental Theorem of Calculus to simplify the value of the area.

2. (10 points) The graph below shows shows the rate of butterfly births in a Monarch butterfly nest over a period of thirty days.



(a) Approximate the area under the curve using a partition of $[0, 30]$ with 3 sub-intervals of equal length and midpoints as sample points. Approximate the function values from the graph.
SOLN: $A \approx 10 \cdot (1 + 85 + 10) = 960$

(b) Explain what the integral $\int_0^{30} f(t)\,dt$ means in terms of the function $r = f(t)$.
SOLN: The integral computes the total number of butterflies born over the 30 day period.

3. (12 points) The speed, $v(t)$, in meters per second of a roller skater as a function of time, t, measured in seconds, is tabulated below:

t (sec)	0	2	4	6	8	10	12
v (m/s)	−3	−2	−0.5	0.5	1	3	3.5

(a) Use these data to compute $R_6 = \sum_{i=1}^{6} v(t_i)\Delta t$ (SOLN:) $= 2(-2 - 0.5 + 0.5 + 1 + 3 + 3.5) = 11$ meters.

(b) Use the data to compute L_6(SOLN:) $= 2(-3 - 2 - 0.5 + 0.5 + 1 + 3) = -2$ meters.

(c) Use the data to compute M_3 (SOLN:) $= 4(-2 + 0.5 + 3) = 6$ meters.

(d) Describe what these sums represent in terms of the roller skater.
SOLN: These sums represent approximations to the displacement the rollerskater's initial position.

4. (15 points) Use the Fundamental Theorem of Calculus Part I to compute the derivative of the given functions:

(a) $g(x) = \int_{-1}^{x} e^{-t^2}\,dt \Rightarrow \boxed{g'(x) = e^{-x^2}}$

(b) $F(x) = \int_{x}^{0} \arctan t^2\,dt \Rightarrow \boxed{F'(x) = -\arctan(x^2)}$

(c) $h(x) = \int_{1}^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz \Rightarrow h'(x) = \frac{du}{dx}\frac{d}{du}\int_{1}^{u} \frac{z^2}{z^4+1}\,dz = \frac{1}{2\sqrt{x}}\frac{u^2}{u^4+1} \boxed{= \frac{1}{2\sqrt{x}}\frac{x}{x^2+1}}$

(d) $p(x) = \int_{\cos x}^{\sin x} t^2\,dt \Rightarrow p'(x) = \frac{d}{dx}\left[\int_{\cos(x)}^{a} t^2\,dt + \int_{a}^{\sin(x)} t^2\,dt\right] = -\frac{du}{dx}\frac{d}{du}\int_{a}^{u} t^2\,dt + \frac{dv}{dx}\frac{d}{dv}\int_{a}^{v} t^2\,dt =$

$\sin(x)u^2 + \cos(x)v^2 = \boxed{\sin(x)\cos^2(x) + \cos(x)\sin^2(x) = \sin x\cos x(\cos x + \sin x)}$

(e) $c(x) = \int_{0}^{1} \cos(t^2)\,dt \Rightarrow c'(x) = 0$

5. (12 points) Consider the integral $\int_{1}^{5} 2x - x^2\,dx$.

(a) Use the definition of the integral to evaluate it as a limit of Riemann sums. Be sure to state what Δx and x_i are. SOLN: $\Delta x = \frac{4}{n}$, $x_i = 1 + \frac{4i}{n}$

$$\int_{1}^{5} 2x - x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left(2\left(1+\frac{4i}{n}\right) - \left(1+\frac{4i}{n}\right)^2\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left(2+\frac{8i}{n} - \left(1+\frac{8i}{n}+\frac{16i^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{n}\sum_{i=1}^{n}1 - \frac{64}{n^3}\sum_{i=1}^{n}i^2\right) = \lim_{n\to\infty}\left(\frac{4}{n}(n) - \frac{64}{n^3}\left(\frac{n(2n+1)(n+1)}{6}\right)\right) = 4 - \frac{64}{3} = -\frac{52}{3} = -17\tfrac{1}{3}$$

(b) Use the Fundamental Theorem of Calculus to verify your result.

SOLN: $\int_{1}^{5} 2x - x^2\,dx = x^2 - \frac{1}{3}x^3\Big|_1^5 = 25 - \frac{125}{3} - 1 + \frac{1}{3} = -\frac{52}{3} = -17\frac{1}{3}$.

6. (12 points) Compute each limit by interpreting it as a definite integral.

(a) $\lim_{n\to\infty}\frac{3}{n}\sum_{i=0}^{n}\cos\left(1+\frac{3i}{n}\right)$

SOLN: $\lim_{n\to\infty}\frac{3}{n}\sum_{i=0}^{n}\cos\left(1+\frac{3i}{n}\right) = \int_{1}^{4}\cos(x)\,dx = \sin(x)\Big|_1^4 = \sin(4) - \sin(1)$

(b) $\lim_{n\to\infty}\frac{4}{n}\sum_{i=0}^{n}\exp\left(1+\frac{4i}{n}\right)$

SOLN: $\lim_{n\to\infty}\frac{4}{n}\sum_{i=0}^{n}\exp\left(1+\frac{4i}{n}\right) = \int_{1}^{5}\exp(x)\,dx = e^5 - e$

7. (12 points) Evaluate each integral.

(a) $\displaystyle\int_0^1 \frac{z^2}{z^3+1}\,dz$

SOLN: Let $u = z^3 + 1$. Then $du = 2z^2\,dz$ and $\displaystyle\int_0^1 \frac{1}{3}\frac{3z^2}{z^3+1}\,dz = \frac{1}{3}\int_1^2 \frac{du}{u} = \frac{1}{3}\ln(2)$

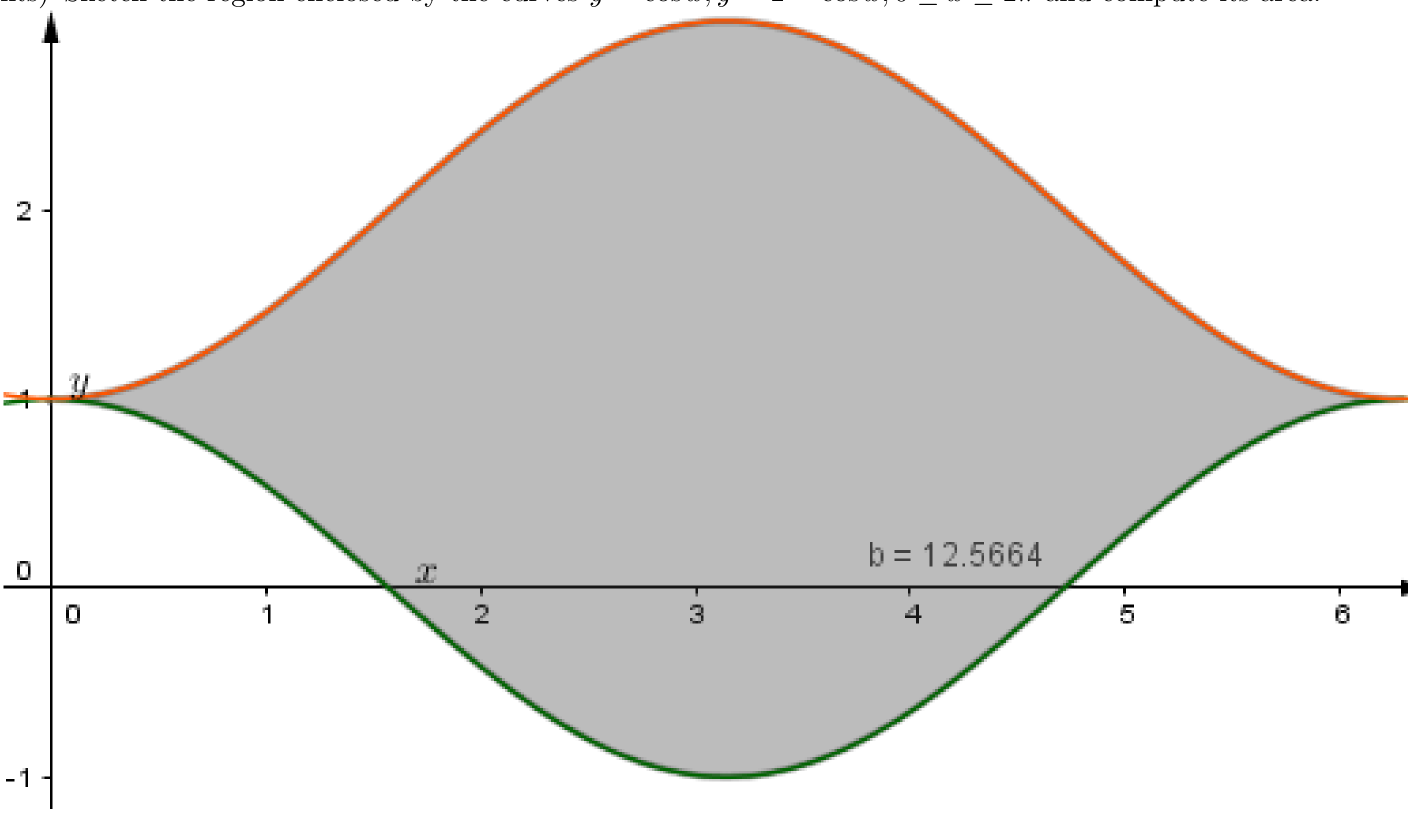
(b) $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$

SOLN: Let $u = a^2 - x^2$. Then $du = -2x\,dx$ and $-\frac{1}{2}\displaystyle\int_0^a -2x\sqrt{a^2-x^2}dx = -\frac{1}{2}\int_{a^2}^0 \sqrt{u}\,du = \frac{1}{3}u^{3/2}\Big|_0^{a^2} = \frac{a^3}{3}$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} x^4\sin(x^3)\,dx$ *Hint*: use a symmetry argument for this.

This is an odd function integrated on an interval centered on 0, therefore the value is 0.

8. (12 points) Sketch the region enclosed by the curves $y = \cos x, y = 2 - \cos x, 0 \le x \le 2\pi$ and compute its area.



SOLN:

$$\text{Area} = \int_0^{2\pi} 2 - \cos(x) - \cos x\,dx = 2x - 3\sin(x)\Big|_0^{2\pi} = 4\pi$$