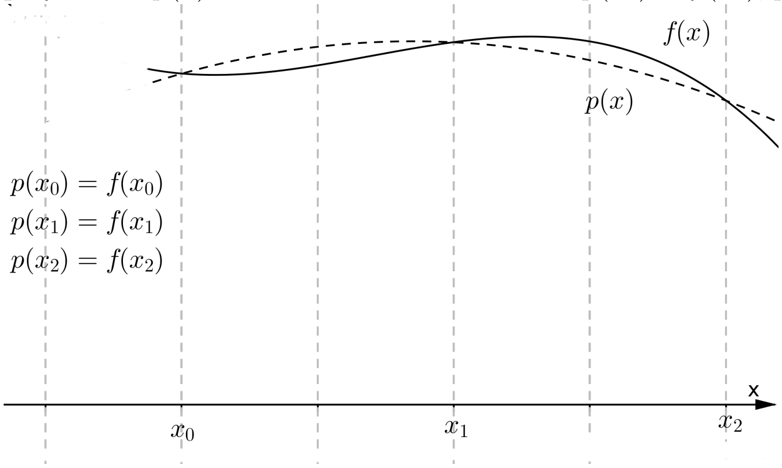
Math 1B Spring 2017–Project

Modifying Simpson's rule. Simpson's rule is a method to approximate the area under a given curve $f(x)$ in a subinterval $[x_0, x_2]$ (composed of two "panels" $[x_0, x_1]$ and $[x_1, x_2]$, (each of width h), by constructing a quadratic polynomial $p(x)$ which fits three constraints: $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$, $p(x_2) = f(x_2)$



The three constraints determine a unique parabola $p(x)$, and $\int_{x_0}^{x_2} p(x)\,dx$ approximates $\int_{x_0}^{x_2} f(x)\,dx$. To improve accuracy, several parabolas may be placed end-to-end in contiguous subintervals, giving Simpson's rule with error term:

$$\int_{x_0}^{x_{2m}} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{2m-2}) + 4f(x_{2m-1}) + f(x_{2m}] - \frac{(x_{2m} - x_0)h^4 f^{(4)}(\mu)}{180}$$

for some $\mu \in (x_0, x_{2m})$. As the number of subintervals increases (and more parabolas are used to approximate $f(x)$), the size of h decreases; this decrease in h reduces the error dramatically since the error is proportional to h^4.

Here is a modification of Simpson's rule. The coefficients are altered, two derivative terms are added, and–most importantly–the error term is improved. The method is analagous to the trapezoidal rule with endpoint correction[1].

Suppose we build a polynomial $q(x)$ by imposing the three constraints of Simpson's rule plus two additional constraints:

$$q(x_0) = f(x_0), q(x_1) = f(x_1), q(x_2) = f(x_2), q'(x_0) = f'(x_0), \text{ and } q'(x_2) = f'(x_2)$$

Think of the two additional constraints as "clamping" the approximating polynomial $q(x)$ to $f(x)$ at the endpoints x_0 and x_2. As we'll see, these five constraints can be used to define a quartic polynomial which, expeanded about the midpoint of the subinterval, has the form

$$q(x) = a_4(x - x_1)^4 + a_3(x - x_1)^3 + a_2(x - x_1)^2 + a_1(x - x_1) + a_0$$

1. (10 points) Simplify

$$\int_{x_0}^{x_2} q(x)\,dx$$

   in terms of h, a_0, a_2, and a_4.

2. (10 points) Use the five constraints to set up a system of five equations in the five unknowns, a_0, a_1, a_2, a_3, a_4, then solve these to find formulas for a_0, a_2, and a_4 in terms of the parameters $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2)$ and $y_0' = f'(x_0), y_2' = f'(x_2)$

[1] Samuel D. Conte and Carl de Boor, Elementary Numerical Analysis, McGraw-Hill, New York, 1980.

3. (10 points) Substitute these values for the coefficients of $q(x)$ to show that

$$\int_{x_0}^{x_2} q(x)\,dx = \frac{h}{15}\left(7f(x_0) + 16f(x_1) + 7f(x_2) + h[f'(x_0) - f'(x_2)]\right)$$

4. (10 points) Simplify

$$\int_{x_0}^{x_{2m}} q(x)\,dx = \int_{x_0}^{x_2} q(x)\,dx + \int_{x_2}^{x_4} q(x)\,dx + \cdots + \int_{x_{2m-2}}^{x_{2m}} q(x)\,dx$$

in terms of the y-values, $y_i = f(x_i), i \in \{0, 1, \ldots 2m\}$ and the slopes $f'(x_0), f'(x_2), \ldots f'(x_{2m})$. How does this compare with Simpson's rule?

5. (30 points) For each of the following integrals, complete a table of errors like this:

n	`Clamped Rule error`	`Simpson's Rule error`
2		
4		
8		
16		

(a) $\int_2^4 \frac{dx}{x}$

(b) $\int_1^5 \ln(x)\,dx$

(c) $\int_0^1 e^{-x^2}\,dx$

For the error in the estimation

$$\int_{x_0}^{x_{2m}} f(x)\,dx \approx \int_{x_0}^{x_{2m}} q(x)\,dx$$

we can prove the following theorem:

Theorem. *If $f^{(6)}(x)$ is continuous on $[x_0, x_{2m}]$, then for some $v \in (x_0, x_{2m})$:*

$$\int_{x_0}^{x_{2m}} f(x)\,dx = \frac{h}{15}\left(7y_0 + 16y_1 + 14y_2 + 16y_3 + 14y_4 + \cdots + 14y_{2m-2} + 16y_{2m-1} + 7y_{2m} + h[f'(x_0) - f'(x_{2m})]\right)$$

$$+\frac{(x_{2m} - x_0)h^6 f^{(6)}(v)}{9450}$$

To determine the error, start by constructing the fifth degree polynomial

$$t(x) = q(x) + k(x - x_0)^2(x - x_1)(x - x_2)^2$$

which has the same integral as $q(x)$ on $[x_0, x_2]$ but allows for sharper estimates.

6. Explain why $\int_{x_0}^{x_2} k(x - x_0)^2(x - x_1)(x - x_2)^2\,dx = 0$

7. (10 points) Show that $t(x)$ satisfies all the constraints of the clamped $q(x)$ and that if we take $k = (f'(x_1) - q'(x_1))/h^4$ then $t'(x_1) = f'(x_1)$.

We leave the completion of the proof for the above theorem for another time.

8. (10 points) It is known that

$$2\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e} \approx 1.1557273497909217.$$

Investigate how the Clamped rule and Simpson's rule compare for approximating this integral as

$$2\int_0^{10{,}000} \frac{\cos x}{1+x^2}\,dx = 2\int_0^{10} \frac{\cos x}{1+x^2}\,dx + 2\int_{10}^{100} \frac{\cos x}{1+x^2}\,dx + 2\int_{100}^{1000} \frac{\cos x}{1+x^2}\,dx + 2\int_{1000}^{10{,}000} \frac{\cos x}{1+x^2}\,dx$$

for $n = 2, 4, 8, 16$.