Math 1B-Spring 2017 Final Exam Solutions

1. Here we will use the infinitesimal approximation, $ds^2 = dx^2 + dy^2$ to estimate the length of the sine curve $y = \sin x$, for $0 \le x \le \pi$.

   (a) Complete the table of values for x and $\frac{ds}{dx} \approx \sqrt{1 + (dy/dx)^2}$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	$\sqrt{2}$	$\frac{\sqrt{7}}{2}$	$\frac{\sqrt{5}}{2}$	1	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{7}}{2}$	$\sqrt{2}$

   (b) Use the tabulated values from (a) to express the right endpoint approximating sum on three subintervals,
   $\int_0^\pi \sqrt{1 + (dy/dx)^2}dx \approx R_3 = \Delta x \sum_{i=1}^{3} \sqrt{1 + \cos^2(x_i)}$ Don't approximate radicals.
   ANS: $\frac{\pi}{3}(\sqrt{5} + \sqrt{2})$

   (c) Use the tabulated values from (a) to express the left endpoint approximating sum on three subintervals,
   $\int_0^\pi \sqrt{1 + (dy/dx)^2}dx \approx L_3 = \Delta x \sum_{i=1}^{3} \sqrt{1 + \cos^2(x_{i-1})}$
   ANS: $\frac{\pi}{3}(\sqrt{5} + \sqrt{2})$

   (d) Find the approximating Simpson sum on six subintervals, S_6 ANS: Clearly $T_3 = \frac{\pi}{3}(\sqrt{5} + \sqrt{2})$. Similarly, $M_3 = \frac{\pi}{3}(\sqrt{7} + 1)$. Thus $S_6 = \frac{2M_3 + T_3}{3} = \frac{\pi}{9}(\sqrt{2} + \sqrt{5} + 2\sqrt{7} + 2)$

2. Find the average value of $f(x) = \frac{1}{1 + x^2}$ on $[0, \pi/4]$. Recall that $f_{\texttt{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$

   ANS: $\frac{1}{\frac{\pi}{4} - 0}\int_0^{\pi/4} \frac{dx}{1 + x^2} = \frac{4}{\pi}\arctan(x)\Big|_0^{\pi/4} = \frac{4}{\pi}\arctan\left(\frac{\pi}{4}\right)$

3. Each given function is continuous on the given interval, so the Mean Value Theorem for integrals applies on the interval. Find all values of c guaranteed by the Mean Value Theorem for Integrals. That is, all c such that, $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$

   (a) $f(x) = \sin x$ on $[0, \pi]$.
   ANS: We want to solve $f(c) = \sin(c) = \frac{1}{\pi}\int_0^\pi \sin(x)dx \Leftrightarrow \sin(c) = \frac{1}{\pi}(1 - \cos(\pi)) \Leftrightarrow \sin(c) = \frac{2}{\pi}$. This equation has two solutions in the interval $[0, \pi]$: $c = \arcsin\left(\frac{2}{\pi}\right)$ and $c = \pi - \arcsin\left(\frac{2}{\pi}\right)$

   (b) $f(x) = 4x^3 + 10x$ on $[0, 3]$.
   ANS: $4c^3 + 10c = \frac{1}{3}(3^4 + 5 \cdot 3^2) \Longleftrightarrow g(c) = 2c^3 + 5c - 21 = 0$. The only possible rational zeros in $[0, 3]$ are $c = 1, \frac{1}{2}, \frac{3}{2}$, but $g(\frac{1}{2}) = -\frac{73}{4}, g(1) = -14, g(\frac{3}{2}) = -\frac{81}{4}$. However, $g(2) = 5$, so there is a zero between $\frac{3}{2}$ and 2. It's approximately $x = 1.8$.

4. Use integration by parts and substitution methods to evaluate the integral. Indicate on the paper what your initial parts are for each.

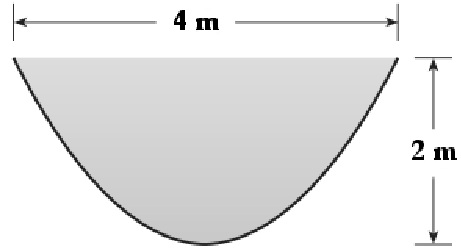
(a) $\displaystyle\int_0^{\ln 2} \frac{x}{e^x}\,dx = -xe^{-x}\Big|_0^{\ln x} + \int_0^{\ln 2} e^{-x}\,dx = \frac{-\ln 2}{2} - \frac{1}{2} + 1$

$u = x$	$dv = e^{-x}dx$
$du = dx$	$v = -e^{-x}$

(b) $\displaystyle\int_0^{\pi/2} \frac{x}{\sec x}\,dx = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx = \frac{\pi}{2} - 1$

$u = x$	$dv = \cos x\,dx$
$du = dx$	$v = \sin x$

5. A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure. Find the hydrostatic force on one end of the trough. Note: the force density of water is $\approx 9800\texttt{N/m}^3$.

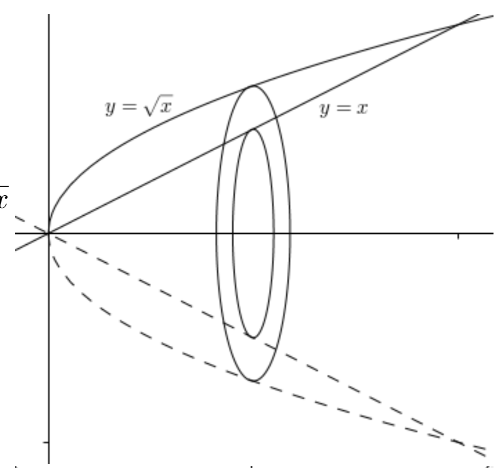


ANS: Let the vertex of the parabola be at the origin of the coordinate system so that the parabola is described by $y = \frac{1}{2}x^2$. Then a point on the parabola has coordinates $(\sqrt{2y}, y)$ so that the width across the parabola at a depth $2-y$ is $w = 2\sqrt{2y}$ given an infinitesimal area of $2\sqrt{2y}dy$. If you multiply the force density by the infinitesimal volume, $dV = 2(2-y)\sqrt{2y}dy$ you get an infinitesimal force you can integrate to get the total hydrostatic force: $F = \int dF = 9810\displaystyle\int_0^2 2(2-y)\sqrt{2y}dy$

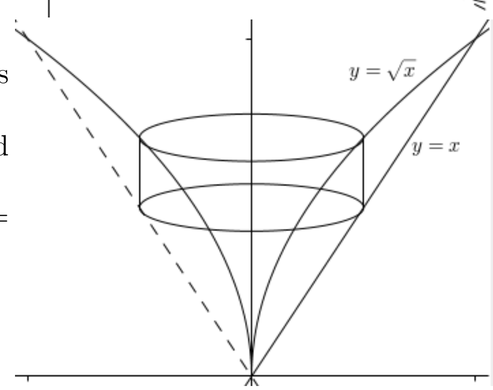
$= 19620\int_0^2 2\sqrt{2}y^{1/2} - \sqrt{2}y^{3/2}\,dy = 19620\sqrt{2}(\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2})\Big|_0^2 = 19620\left(\frac{16}{3} - \frac{16}{5}\right) = 41856$ Newtons.

6. Consider the region bounded by $y = x$ and $y = \sqrt{x}$

(a) Find the volume generated by rotating this region about the x-axis using the washer method.
ANS: As illustrated in the diagram at right, The big radius is $R = \sqrt{x}$ and the little radius is $r = x$, so the volume is $V = \int dV$
$= \pi\displaystyle\int_0^1 (\sqrt{x})^2 - x^2\,dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$



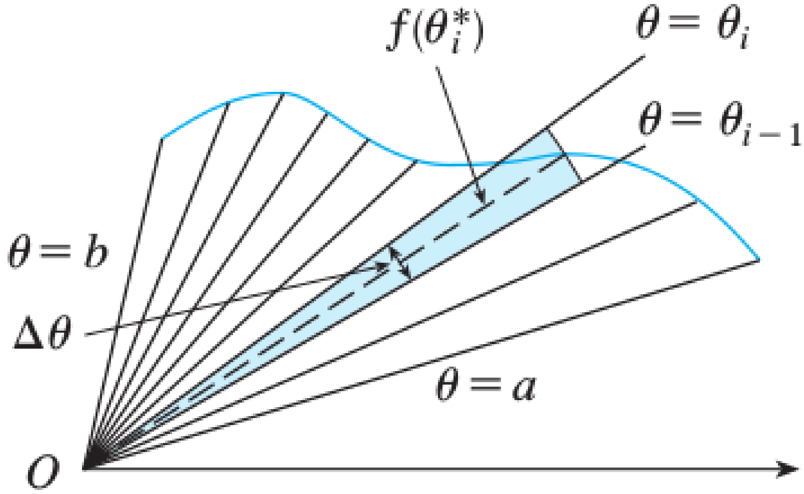
(b) Find the volume generated by rotating this region about the y-axis using the shell method (different volume!)
ANS: As illustrated in the diagram at right, the radius of a shell is x and the height is $\sqrt{x} - x$ so the volume is $V = \int dV = 2\pi\displaystyle\int_0^1 x(\sqrt{x} - x)\,dx =$
$2\pi(\left(\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right)\Big|_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}$

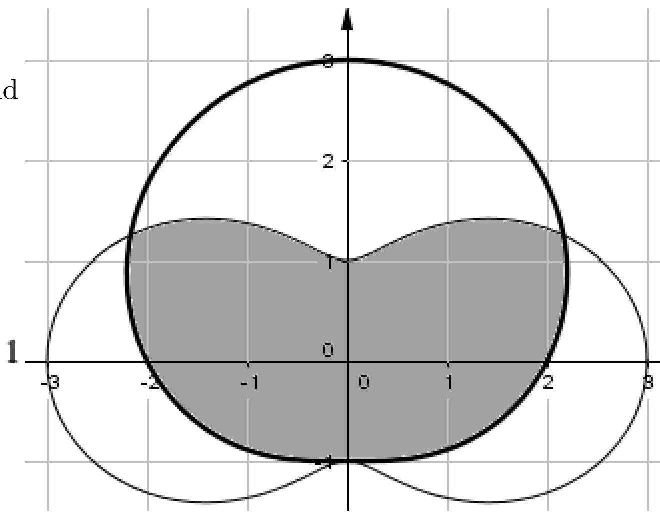


7. Consider the problem of finding area by integrating with polar coordinates.

(a) Explain why the area enclosed by a curve $r = f(\theta)$ for $\theta_1 \le \theta \le \theta_2$ is given by $\displaystyle\int_{\theta_1}^{\theta_2} \frac{(f(\theta))^2}{2}\, d\theta$. Draw a diagram and describe the relevant formula.

ANS: The book's diagram is good:





The formula for the area of a sector of radius $f(\theta)$ and central angle $\Delta\theta$ is $\dfrac{(f(\theta))^2\Delta\theta}{2}$

(b) Find the area of the region inside the polar curve $r = 2 + \cos 2\theta$ and inside the curve $r = 2 + \sin\theta$, as shaded in the diagram.

ANS: First we need find the point of intersection of the curves. That is, where $2 + \cos(2\theta) = 2 + \sin\theta \Leftrightarrow$ $\Leftrightarrow 1 - 2\sin^2\theta = \sin\theta \Leftrightarrow 2\sin^2\theta + \sin\theta - 1 = 0 \Leftrightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$ So the point of intersection in the first quadrant is where $\sin\theta = \frac{1}{2} \Leftarrow \theta = \dfrac{\pi}{6}$. Using symmetry, we can compute the total area as twice the area for $\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{6}$ (where the region is bounded by $r = 2 + \sin\theta$) and twice the area for $\dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2}$:

$$\texttt{Area} = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}(2+\sin\theta)^2\, d\theta + 2\int_{\pi/6}^{\pi/2} \frac{1}{2}(2+\cos(2\theta))^2\, d\theta = \int_{-\pi/2}^{\pi/6} 4 + 4\sin\theta + \sin^2\theta\, d\theta + \int_{\pi/6}^{\pi/2} 4 + 4\cos(2\theta) + \cos^2(2\theta)\, d\theta$$

$$= \left(4\theta - 4\cos\theta + \frac{\theta - \sin\theta\cos\theta}{2}\right)\Big|_{-\pi/2}^{\pi/6} + \left(4\theta + 2\sin(2\theta) + \frac{2\theta + \sin 2\theta\cos 2\theta}{4}\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \frac{9}{2}\theta - 4\cos\theta - \frac{\sin 2\theta}{4}\Big|_{-\pi/2}^{\pi/6} + \frac{9}{2}\theta + 2\sin(2\theta) + \frac{\sin 4\theta}{8}\Big|_{\pi/6}^{\pi/2}$$

$$= \left(\frac{3\pi}{4} - 2\sqrt{3} - \frac{\sqrt{3}}{8}\right) - \left(-\frac{9}{4}\pi + 0\right) + \left(\frac{9\pi}{4} + 0\right) - \left(\frac{3\pi}{4} + \sqrt{3} + \frac{\sqrt{3}}{16}\right)$$

$$= \frac{6 + 18 + 18 - 6}{8}\pi - \frac{32 + 2 + 16 + 1}{16}\sqrt{3} = \frac{9\pi}{2} - \frac{51\sqrt{3}}{16}$$

That was much calculation, each step dependent on accurate completion of the previous step. Let's check that using Sagemath (Now CoCalc?):

```
var('t')
integral(4+4*cos(2*t)+(cos(2*t))^2,t,pi/6,pi/2)+integral(4+4*sin(t)+(sin(t))^2,t,-pi/2,pi/6)

9/2*pi - 51/16*sqrt(3)
```

8. Use a Maclaurin series to approximate $\sqrt{1.2}$ to the nearest millionth (10^{-6}). Recall that $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$

ANS: $\sqrt{1.2} = (1+\frac{1}{5})^{1/2} = f(\frac{1}{5})$ where $f(\frac{1}{5}) = (1+\frac{1}{5})^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (\frac{1}{5})^n$

Thus, $\sqrt{1.2} \approx 1 + \frac{1}{2}\cdot\frac{1}{5} - \frac{1}{8}\cdot\frac{1}{5^2} + \frac{1}{16}\cdot\frac{1}{5^3} - \frac{5}{128}\cdot\frac{1}{5^4} + \frac{7}{256}\cdot\frac{1}{5^5}$.

At this stage one can pause to observe this is an alternating series and so the first neglected term is an upper bound on the error. $\frac{7}{2^8}\cdot\frac{1}{5^5} = \frac{7}{800000}$ is not quite there yet, so we get the next term, which is $\dfrac{21}{2^{10}}\cdot\dfrac{1}{5^6} = \dfrac{21}{16\times 10^6}$.

That'll do. So $\sqrt{1.2} \approx 1 + \frac{1}{2}\cdot\frac{1}{5} - \frac{1}{8}\cdot\frac{1}{5^2} + \frac{1}{16}\cdot\frac{1}{5^3} - \frac{5}{128}\cdot\frac{1}{5^4} + \frac{7}{256}\cdot\frac{1}{5^5} - \dfrac{21}{16\times 10^6}$.

With the aid of a calculator - oh, heck, with Python, we can write a little script:

```python
import scipy.special

sum = 0
pow5 = 1
for n in range(7):
    sum += scipy.special.binom(0.5, n)/pow5
    pow5 *= 5
    print(sum)
```

which produces this output:

```
1.0
1.1
1.095
1.0955
1.0954375
1.09544625
1.0954449375
```

A desk calculator computes $\sqrt{1.2} \approx 1.09544511501$, which differs from the 7-term approximation by about 1.8e-7.

9. Use the Taylor series, $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ to express $f(x) = x^3 + x + 1$ as a sum of multiples of powers of $x-1$.

ANS: $a_0 = f(1) = 3, a_1 = f'(1) = 4, a_2 = f''(1)/2 = 3$ and $a_3 = 6/3! = 1$, so $f(x) = 3+4(x-1)+3(x-1)^2+(x-1)^3$