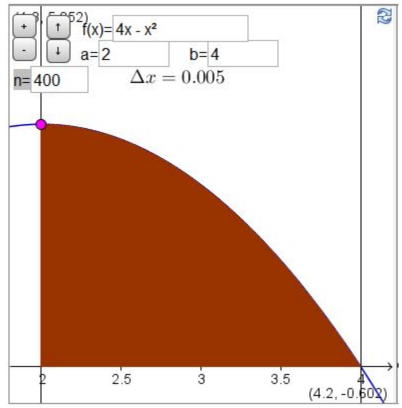
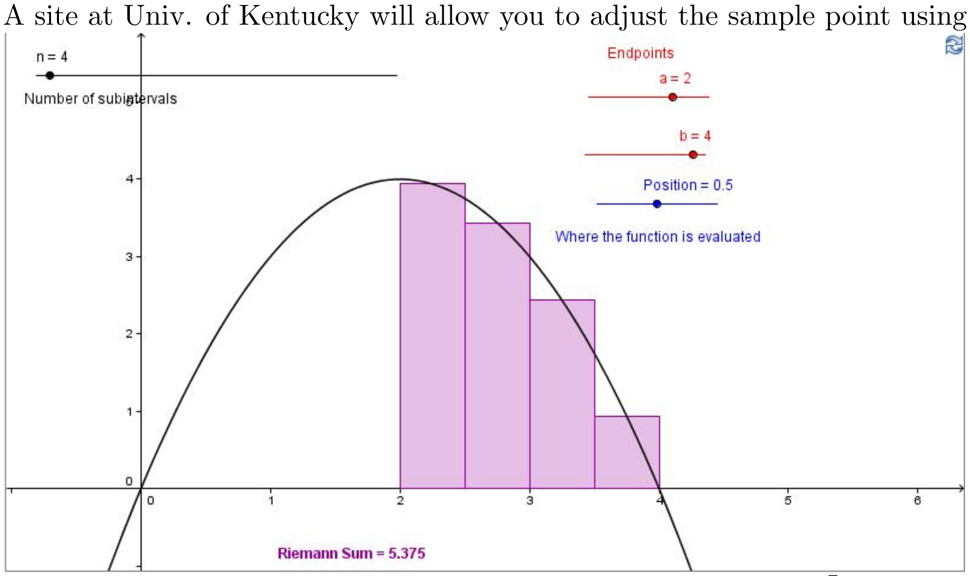
Math 1B §5.1 – §6.3 Test Solutions, Fall '14

1. (20 points) Consider the function $f(x) = 4x - x^2$ on the interval $2 \le x \le 4$.

   (a) Draw a diagram illustrating the area of the region bounded by this curve and the x-axis.
   At math insight you can enter the info for this problem to produce a diagram like that shown at right. To fill in the entire area, just choose a very large value for n (I have $n = 400$ here):



   (b) Approximate this area using a partition with 4 intervals of equal length and midpoints as sample points.
   A site at Univ. of Kentucky will allow you to adjust the sample point using a slider bar:



   The midpoint sum is $\sum_{i=1}^{4} \Delta x \cdot f(x_i^*) = \sum_{i=1}^{4} \frac{1}{2} \cdot f\left(\frac{7}{4} + \frac{i}{2}\right) = \frac{1}{2} \sum_{i=1}^{4} \left[\cdot 4\left(\frac{7}{4} + \frac{i}{2}\right) - \left(\frac{7}{4} + \frac{i}{2}\right)^2 \right]$

   To be sure, let's tabulate the (x, y) values at the midpoints:

x	$\frac{9}{4}$	$\frac{11}{4}$	$\frac{13}{4}$	$\frac{15}{4}$
y	$\frac{63}{16}$	$\frac{55}{16}$	$\frac{39}{16}$	$\frac{15}{16}$

   $= \frac{1}{2}\left(\frac{63}{16} + \frac{55}{16} + \frac{39}{16} + \frac{15}{16}\right) = \frac{43}{8} = 5.375$

   (c) Use the definition of the definite integral to compute the area as the limit of Riemann sums.
   2 Note that the definition is *not* the Fundamental Theorem of Calculus.

   $$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \cdot f\left(2 + \frac{2i}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \cdot \left[4\left(2 + \frac{2i}{n}\right) - \left(2 + \frac{2i}{n}\right)^2\right] =$$

   $$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(4 - \frac{4i^2}{n^2}\right) = \lim_{n\to\infty} 8 - \frac{8}{n^3} \sum_{i=1}^{n} i^2 = 8 - \lim_{n\to\infty} \frac{8n(2n+1)(n+1)}{6n^3} = 8 - \frac{8}{3} = \frac{16}{3}$$

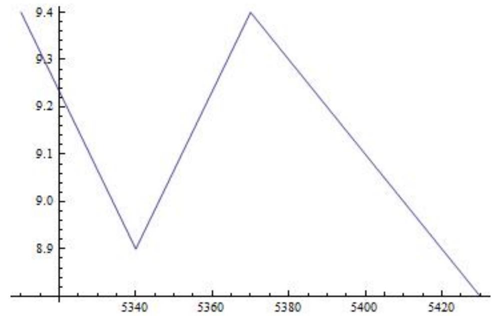
   (d) Based on the results of parts (b) and (c) above, is (b) an overestimate or an underestimate of the area? What is the error?
   SOLN: The error is the difference between the estimate and the true value $= \frac{43}{8} - \frac{16}{3} = \frac{1}{24}$, which shows (since it's positive) that the midpoint sum is an overestimate. This makes sense since the curve is concave down. Why?

2. (15 points) The table at right below shows shows the total population of Finland (in 1000's of people) and the carbon dioxide (CO_2) emissions per person over the years 2008–2012.

(a) Plot $y = f(x)$ where y = tons of CO_2/person as a function of x = population (in 1000's) on the domain $5310 \le x \le 5430$ and range $8.8 \le y \le 9.4$. Scale your plot to focus on this domain and range and connect the dots.

(b) Explain what the integral $\int_{5310}^{5430} f(x)\,dx$ measures.
What are the units of measure?
SOLN: I used Mathematica to produce the plot:

$$\text{ListLinePlot}\left[\begin{pmatrix} 5310 & 9.4 \\ 5340 & 8.9 \\ 5370 & 9.4 \\ 5400 & 9.1 \\ 5430 & 8.8 \end{pmatrix}\right]$$



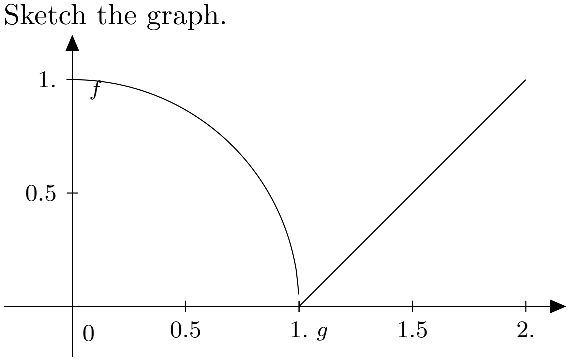
The integral will measure the net change in tons of CO_2 emissions as the Finnish population increases from 5,310,000 to 5,430,000.

(c) Use 4 subintervals of equal length to approximate this integral sampling at right end-points.
$\int_{5310}^{5430} f(x)\,dx \approx 30(8.9+9.4+9.1+8.8) = 30\cdot 36.2 =$ 1086 thousands of tons of increased carbon dioxide emissions.

3. (15 points) Consider this piecewise function :

$$f(x) = \begin{cases} \sqrt{1-x^2} & : 0 \le x \le 1 \\ x-1 & : 1 \le x \le 2 \end{cases}$$

(a) Sketch the graph.



(b) Find the area below the function and above the x-axis by using familiar area formulas from geometry.
SOLN: This is just a quarter of the unit circle plus a right isosceles triangle of side 1:
Area $= \frac{\pi}{4} + \frac{1}{2}$ square units.

4. (11 points) If $f(x)$ is a continuous function, prove that $\int_0^1 f(4x+1)dx = \frac{1}{4}\int_1^5 f(x)dx$

SOLN: Let $u = 4x+1$. Then $du = 4dx$. As $0 \le x \le 1$, $1 \le u \le 5$ so the substitution yields
$\int_0^1 f(4x+1)dx = \frac{1}{4}\int_1^5 f(u)du$. Since the symbol used for the variable of integration is immaterial, we have that

$$\int_0^1 f(4x+1)dx = \frac{1}{4}\int_1^5 f(u)du = \frac{1}{4}\int_1^5 f(x)dx$$

5. (18 points) Consider the integral function $G(x) = \int_a^x te^{t^2}dt$

(a) Find $\frac{d}{dx}G(x)$

SOLN: $\frac{d}{dx}\int_a^x te^{t^2}dt = xe^{x^2}$

(b) Simplify $G(x)$ in terms of a.

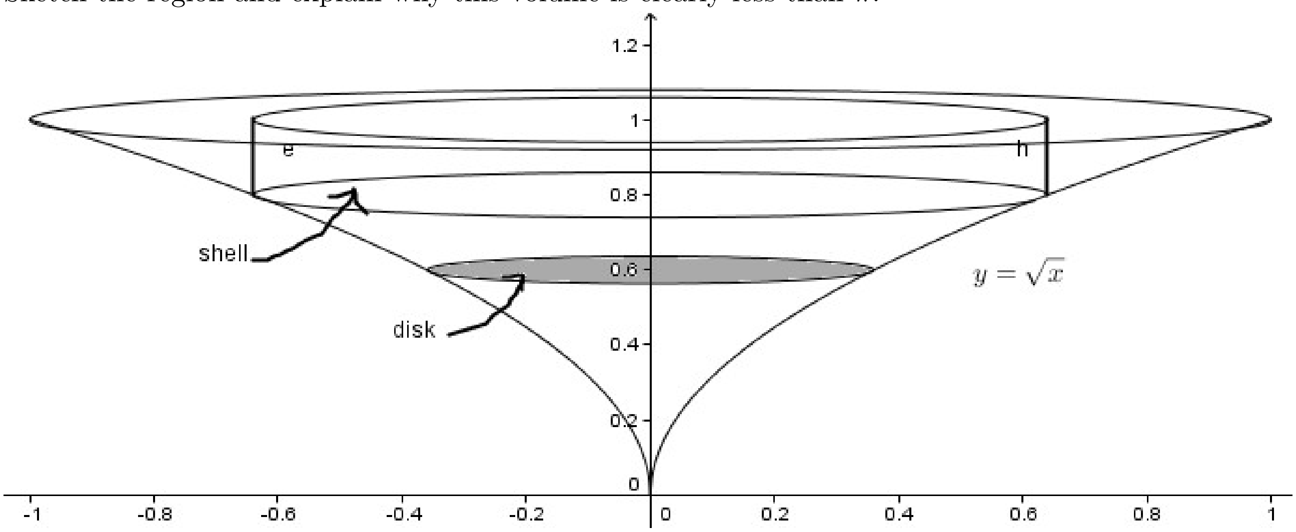
SOLN: Substitute $u = t^2 \Rightarrow du = 2tdt$ and note that $a \le t \le x \Rightarrow a^2 \le u \le t^2 \Rightarrow G(x) = \frac{1}{2}\int_{a^2}^{x^2} e^u du = \frac{1}{2}\left(e^{x^2} - e^{a^2}\right)$

(c) For what value of a is $G(x) = \frac{1}{2}\left(e^{x^2} - e\right)$?

$a \pm 1$ To be sure, note that $f(t) = te^{t^2}$ is an odd integrand, so $\int_{-1}^{1} te^{t^2}dt = 0$

6. (21 points) Consider volume generated when the region in the first quadrant bounded by $\sqrt{x} \le y \le 1$ and $0 \le x \le 1$ is revolved about the line y-axis.

(a) Sketch the region and explain why this volume is clearly less than π.



(b) Find the volume using the method of washers.

SOLN: $\pi\int_0^1 \left(y^2\right)^2 dy = \frac{\pi}{5}y^5\Big|_0^1 = \frac{\pi}{5}$

(c) Find the volume using the method of shells.

$2\pi\int_0^1 x\left(1 - \sqrt{x}\right)dx = 2\pi\frac{x^2}{2} - \frac{4}{5}x^{5/2}\Big|_0^1 = \frac{\pi}{5}$

7. (0 points) Consider volume generated when the region in the first quadrant bounded by $2x \le y \le 4x - x^2$ is revolved about the x-axis.

(a) By washers.

SOLN: $V = \int dV = \pi\int_0^2 R^2 - r^2 dx = \pi\int_0^2 (4x-x^2)^2 - (2x)^2 dx = \pi\int_0^2 12x^2 - 8x^3 + x^4 dx = \pi\left(4x^3 - 2x^4 + \frac{x^5}{5}\right)\Big|_0^2 = \frac{32\pi}{5}$

(b) Find the volume using the method of shells.

$V = \int dV = 2\pi \int\limits_0^4 rhdy = 2\pi \int\limits_0^4 y\left(\frac{y}{2} - 2 + \sqrt{4-y}\right) dy = 2\pi \int\limits_0^4 \frac{y^2}{2} - 2y + y\sqrt{4-y}dy = 2\pi \left[\frac{y^3}{6} - y^2\right]\Big|_0^4 +$
$2\pi \int\limits_0^4 4u^{1/2} - u^{3/2}du = \frac{-32\pi}{3} + 2\pi\left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^4 = \frac{-32\pi}{3} + 2\pi\left[\frac{64}{3} - \frac{64}{5}\right] = \frac{32\pi}{5}$