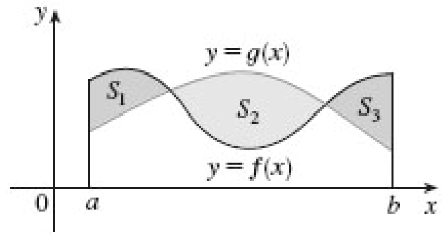
Math 1B – Calculus II – Chapter 6 Problems – Fall '11 Name________________________________

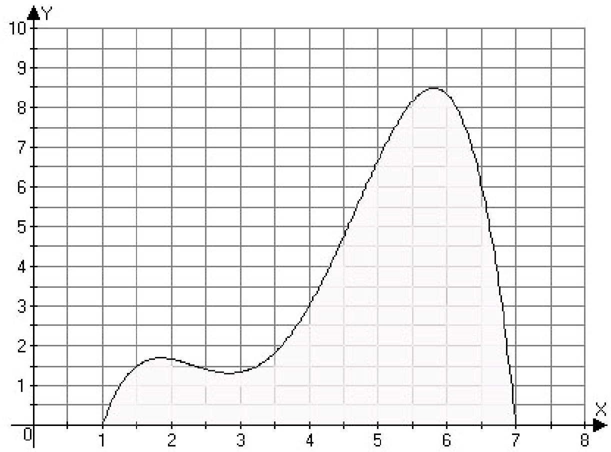
Show your work for credit. Write all responses on separate paper. Do not use a calculator.

1. The diagram at right (from the text) shows a situation where functions cross on the interval $[a,b]$. In a situation like this, to find the area between the curves, you need to split the integral up into pieces. Set up integrals to find the area between the curves $y = -4x^2 + 12x - 6$ and $y = x^3 - 3x^2 + 2x + 2$ on $[0,3]$.



2. If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 3$ to set up a sum to estimate to the nearest tenth the volume of the solid. You don't need to evaluate the sum.

3. If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 3$ to set up a sum to estimate to the nearest tenth the volume of the solid. You don't need to evaluate the sum

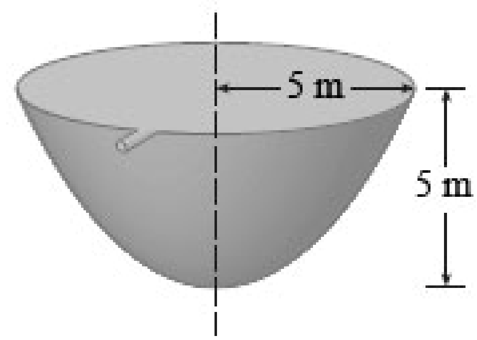


4. Set up an integral to compute the volume of the solid whose base is an equilateral triangle of height $\sqrt{3}$ and with semi-circular cross-sections perpendicular to the base. Assume that the diameter of the largest semi-circle is the altitude of the triangle.

5. Set up integrals (you do not need to evaluate these) to find the volume of the solid generated by revolving the region bounded by $y = 2x$ and $y = 2 - 2(x-1)^2$ about the line $x = -1$
   a. using the washer method.
   b. using the shell method.

6. If the work required to stretch a spring 1 m beyond its natural length is 12 J, how much work is needed to stretch it 2 m? Hint: First use Hooke's law to compute the spring constant k in the model $F(x) = kx$.

7. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
   a. If its height is 5 m and the radius at the top is 5 m, find the work required to pump the water out of the tank.
   b. Write an equation you could solve to compute the water level in the tank after 915600π J of work is done pumping the water out.



8. Consider the function $f(x) = x^2$.
   a. Find the average value of the function over the interval [0,b].
   b. What condition does the function satisfy that the Mean Value Theorem for Integrals allows us to conclude that there is some c between 0 and b so that $f(c)$ = the average value?

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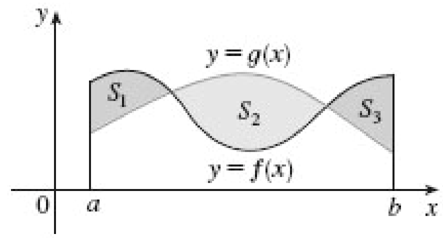
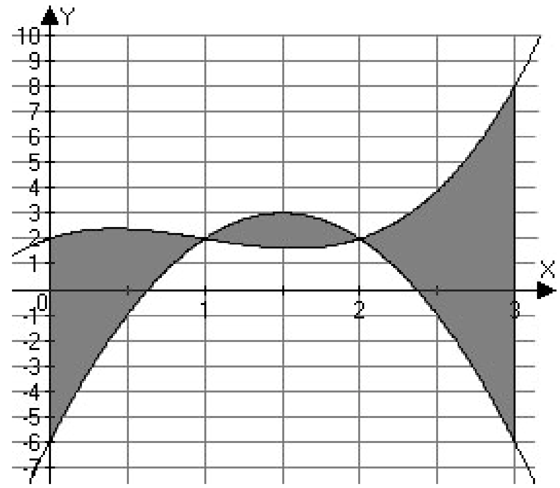
1. The diagram at right (from the text) shows a situation where functions cross on the interval $[a,b]$. In a situation like this, to find the area between the curves, you need to split the integral up into pieces. Set up integrals to find the area between the curves $y = -4x^2 + 12x - 6$ and $y = x^3 - 3x^2 + 2x + 2$ on $[0,3]$.





SOLN: To find where these functions intersect, we solve

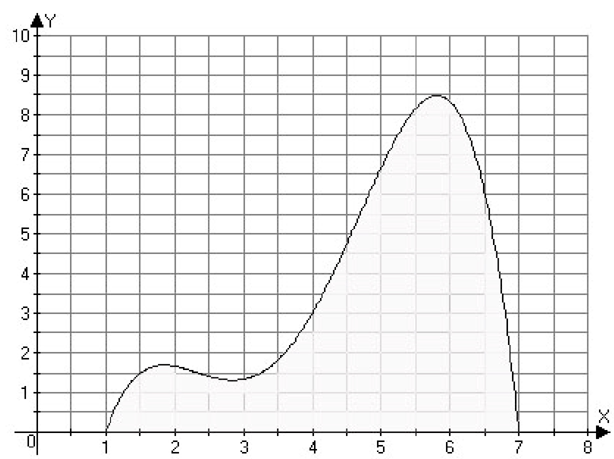
$$f(x) = g(x)$$
$$-4x^2 + 12x - 6 = x^3 - 3x^2 + 2x + 2 \leftrightarrow$$
$$x^3 + x^2 - 10x + 8 = (x-1)(x-2)(x+4) = 0.$$

Tabulating a few values we have

x	0	1/2	1	3/2	2	5/2	3
$f(x)$	–6	–1	2	3	2	–1	–6
$g(x)$	2	29/8	2	13/8	2	31/8	8

Thus the area between the curves is

$$\int_0^1 x^3 + x^2 - 10x + 8dx + \int_1^2 -x^3 - x^2 + 10x - 8dx$$
$$+\int_2^3 x^3 + x^2 - 10x + 8dx$$



2. If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 3$ to set up a sum to estimate to the nearest tenth the volume of the solid. You don’t need to evaluate the sum. Here $\Delta x = 2$

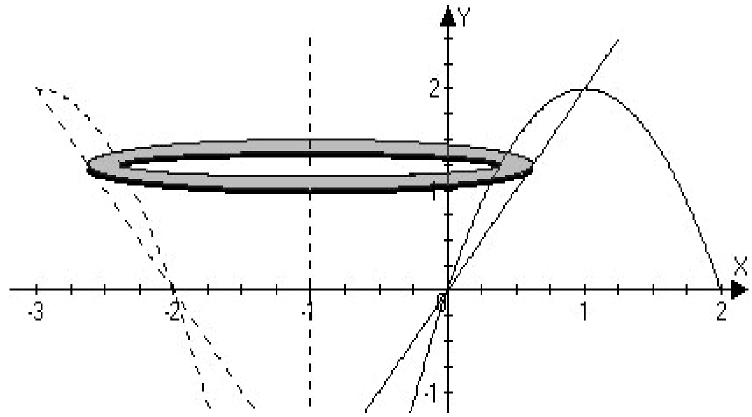
   SOLN: $\pi \sum_{i=1}^{3} r^2 dx \approx 2\pi\left(1.7^2 + 3.0^2 + 8.3^2\right)$

3. If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 3$ to set up a sum to estimate to the nearest tenth the volume of the solid. You don’t need to evaluate the sum. SOLN:

$$2\pi \sum_{i=1}^{3} rhdx \approx 4\pi\left(2(1.7) + 4(3.0) + 6(8.3)\right)$$

4. Set up an integral to compute the volume of the solid whose base is an equilateral triangle of height $\sqrt{3}$ and with semi-circular cross-sections perpendicular to the base. Assume that the diameter of the largest semi-circle is the altitude of the triangle.
   SOLN: Over half the triangle, the diameter will go like $2r = \sqrt{3}(1 - x)$. So the radius goes like $r = \sqrt{3}\,(1 - x)/2$ and the area of the semicircle is $A(x) = \pi r^2/2 = 3\pi(1 - x)^2/8$ since this gives only half the area, we double it to get $V = \frac{3\pi}{4}\int_0^1 (1-x)^2\, dx = -\frac{\pi(1-x)^3}{4}\Bigg|_0^1 = \frac{\pi}{4}$

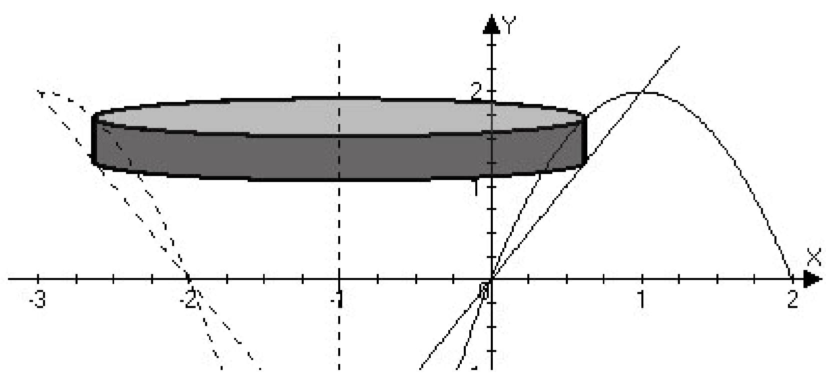
5. Set up integrals (you do not need to evaluate these) to find the volume of the solid generated by revolving the region bounded by $y = 2x$ and $y = 2 - 2(x-1)^2$ about the line $x = -1$



a. using the washer method.
SOLN: $dV = \pi(R^2 - r^2)$ where
$R = 1 + y/2$ and $r = 1 - \sqrt{(1 - y/2)}$

So V = $\pi\int_0^2 \left(\frac{y}{2}+1\right)^2 - \left(2-\sqrt{1-\frac{y}{2}}\right)^2 dy$ This is sufficient to answer the question. A gratuitous evaluation proceeds by substituting $u = 1 - \frac{y}{2}$, yeilding

$$V = -2\pi\int_1^0 (2-u)^2 - \left(2-\sqrt{u}\right)^2 du$$
$$= 2\pi\int_0^1 u^2 - 4u + 4 - \left(4 - 4\sqrt{u} + u\right) du$$
$$= 2\pi\int_0^1 u^2 - 5u + 4\sqrt{u}du$$
$$= 2\pi\left(\frac{u^3}{3} - \frac{5}{2}u^2 + \frac{8}{3}u^{3/2}\right)_0^1 = 2\pi\left(\frac{1}{3} - \frac{5}{2} + \frac{8}{3}\right) = 2\pi\left(\frac{2-15+16}{6}\right) = \pi$$



b. using the shell method.
SOLN: $dV = 2\pi rh$ where
$h = 2 - 2(x-1)^2 - 2x$ and $r = 1 + x$

So V = $4\pi\int_0^1 (1+x)(2-2(x-1)^2 - 2x)dx$ This answers the question. To be more sure, if we simplify:

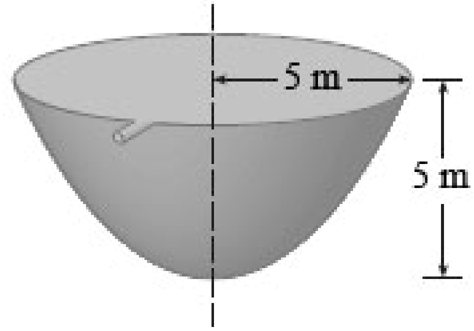
$= 4\pi\int_0^1 \left(x - x^3\right) dx = \pi$ So the two methods get the same result.

6. If the work required to stretch a spring 1 m beyond its natural length is 12 J, how much work is needed to stretch it 2 m? Hint: First use Hooke's law to compute the spring constant k in the model $F(x) = kx$.

SOLN: The work to stretch the spring 1m is $W = \int dW = \int_0^1 kx dx = \frac{k}{2} = 12 \Rightarrow k = 24$

Thus, the work to stretch this spring 3 meters from equilibrium is $W = 24\int_0^2 x dx = 12x^2\Big|_0^2 = 48$

7. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.



a. If its height is 5 m and the radius at the top is 5 m, find the work required to pump the water out of the tank.
SOLN: Placing the origin of the coordinate system at the vertex of the parabola the equation of the parabola is $y = x^2/5$ and so the radius of a circular cross-section at y is $x = \sqrt{(5y)}$ and the area of a circular cross-section is $A = 5\pi y$ so an infinitesimal volume at an equipotential in the gravitational field is $dV = Ady = 5\pi y dy$ so the an element of work is $dW = 9810(5 - y)dV = 9810(5 - y)5\pi y dy$ and the total work is

$$W = 9810(5\pi)\int_0^5 5y - y^2 dy = 9810(5\pi)\left(\frac{5y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^5 = 9810(5\pi)\frac{5^3}{6} = 1635(5^4)\pi$$

b. Write an equation you could solve to compute the water level in the tank after 915600π J of work is done pumping the water out.

SOLN: Solve for h: $W = 9810(5\pi)\int_h^5 5y - y^2 dy = 915600\pi$

8. Consider the function $f(x) = x^2$.

a. Find the average value of the function over the interval $[0,b]$.

SOLN: $f_{\text{avg}} = \frac{1}{b-0}\int_0^b x^2 dx = \frac{b^3}{3b} = \frac{b^2}{3}$

b. What condition does the function satisfy that the Mean Value Theorem for Integrals allows us to conclude that there is some c between 0 and b so that $f(c)$ = the average value?
SOLN: That f is continuous on $[0,b]$