Math 1B—Calculus II – Test 4 – Spring '10 Name________________________________

Write all responses on separate paper. Show your work for credit. Don't use a calculator.

1. Find the area in the xy–plane enclosed by the x-axis and the curve described by the parametric equations $x = 1 + e^t$ and $y = 3t - t^2$.

2. Consider the parametric equations describing a hyperbola: $\boxed{\begin{array}{l} x = 1 + 2\sec t \\ y = 3 + 4\tan t \end{array}}$

   a. Write the equation for the hyperbola in standard form by specifying values for a, b, h, and k in the formula $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b} = 1$

      Hint: Recall the identity $\sec^2 t - \tan^2 t = 1$

   b. Find a value of t so that the tangent line at $(x(t), y(t))$ has slope = 4.

   c. Find the value of $\left|\dfrac{d^2y}{dx^2}\right|$ where $x = 5$.

3. Find the area of the loop formed by $\begin{array}{l} x = t^2 + 2 \\ y = t\left(t^2 - 9\right) \end{array}$

   Hint: The loop closes where the curve intersects itself: where we can find two different parameter values, $t_1 \neq t_2$, such that $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$.

4. Find the area that lies inside the curve $r = 2 + \cos\theta$ and outside $r = \cos(2\theta)$.

5. The area of the surface generated by rotating the polar curve $r = f(\theta)$ for $a \leq \theta \leq b$ about the polar axis (the x-axis) is $S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta$. Use this formula to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

6. List the first five terms of the sequence $a_1 = 2$, $a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$.

7. Determine whether or not the series $\sum_{n=0}^{\infty} \dfrac{\left(\sqrt{2}\right)^{n+1}}{2^n}$ converges. If it converges, find its sum.

8. Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{4n^2+32n+63}$ is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

9. Find a value of c such that $\sum_{n=0}^{\infty} e^{cn} = 3$

10. Give a convincing argument as to whether the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent or divergent.

11. Use the integral inequality $s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$

to estimate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to the nearest thousandth.

12. Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converges or diverges.

13. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1+a_n)$ is convergent.

14. Test the series $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{(n-2)\pi}{2n}\right)$ for convergence or divergence.

15. How many terms are needed to approximate the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ to the nearest ten thousandth?

Math 1B—Calculus II – Test 4 Solutions – Spring '10

1. Find the area in the xy–plane enclosed by the x-axis and the curve described by the parametric equations $x = 1 + e^t$ and $y = 3t - t^2$.

$$\int_0^3 y\frac{dx}{dt}dt = \int_0^3 \left(3t-t^2\right)e^t dt \quad \boxed{\begin{array}{ll} u = 3t-t^2 & dv = e^t dt \\ du = (3-2t)\,dt & v = e^t \end{array}}$$

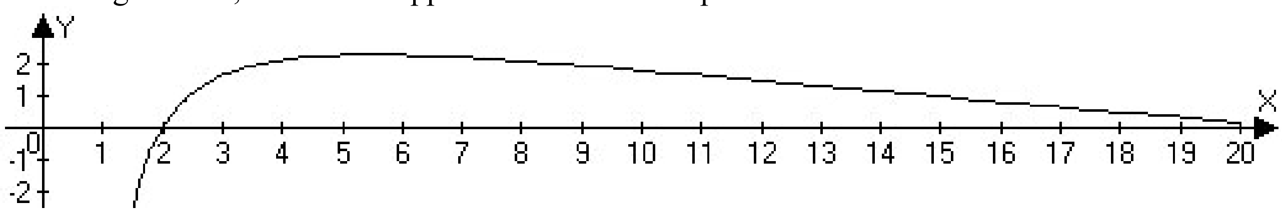
SOLN: $y > 0$ on (0,3) so the area is

$$= \left(3t-t^2\right)e^t\Big|_0^3 - \int_0^3 (3-2t)e^t dt \quad \boxed{\begin{array}{ll} u = 3-2t & dv = e^t dt \\ du = 2dt & v = e^t \end{array}}$$

$$= \left(3t-t^2\right)e^t - (3-2t)e^t\Big|_0^3 - 2\int_0^3 e^t dt$$

$$= \left(-t^2+5t-5\right)e^t\Big|_0^3 = e^3 + 5 \approx 25.1$$

As the graph below indicates, the area is about 18 units long and has an average height of the something like 1.4, so the 25.1 approximation is in the pocket:



2. Consider the parametric equations describing a hyperbola: $\boxed{\begin{array}{l} x = 1+2\sec t \\ y = 3+4\tan t \end{array}}$

   a. Write the equation for the hyperbola in standard form by specifying values for a, b, h, and k in the formula $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b} = 1$

   Hint: Recall the identity $\sec^2 t - \tan^2 t = 1$

   SOLN: $\dfrac{(x-1)^2}{4} - \dfrac{(y-3)^2}{16} = 1$

   b. Find a value of t so that the tangent line at $(x(t), y(t))$ has slope = 4.

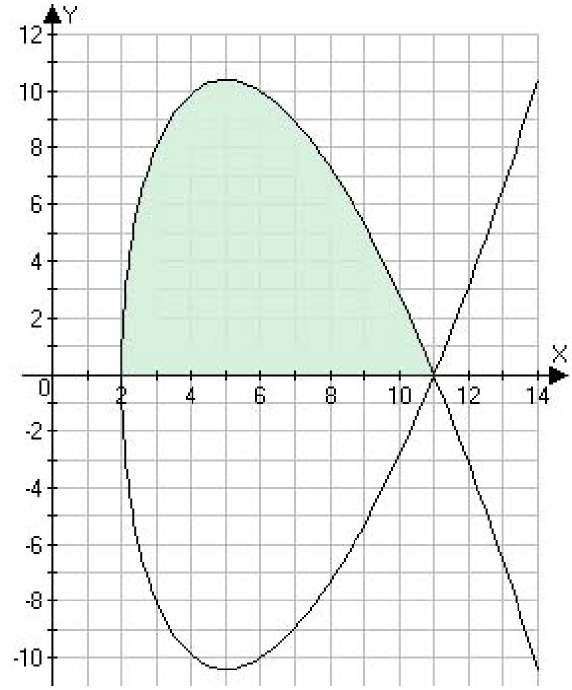
   SOLN: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\sec^2 t}{2\sec t\tan t} = \dfrac{2\sec t}{\tan t} = \dfrac{2}{\sin t} = 4 \Leftrightarrow \sin t = \dfrac{1}{2} \Leftrightarrow t = \dfrac{\pi}{2} \pm \dfrac{\pi}{3} + 2\pi k$

   c. Find the value of $\left|\dfrac{d^2y}{dx^2}\right|$ where $x = 5$.

   SOLN: If $x = 5$ then sec $t = 2$ so cos $t = ½$.

$$\left|\frac{d^2y}{dx^2}\right| = \left|\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}\right| = \left|\frac{\frac{d}{dt}\left(\frac{2}{\sin t}\right)}{2\sec t\tan t}\right| = \left|\frac{-2\csc t\cot t}{2\sec t\tan t}\right| = \left|\frac{\cos^3 t}{\sin^3 t}\right| = \left|\frac{\cos t}{\sqrt{1-\cos^2 t}}\right|^3 = \left(\frac{1/2}{\sqrt{3}/2}\right)^3 = \frac{\sqrt{3}}{9}$$

3. Find the area of the loop formed by $\begin{matrix} x = t^2 + 2 \\ y = t\left(t^2 - 9\right) \end{matrix}$



Hint: The loop closes where the curve intersects itself: where we can find two different parameter values, $t_1 \neq t_2$, such that $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$.

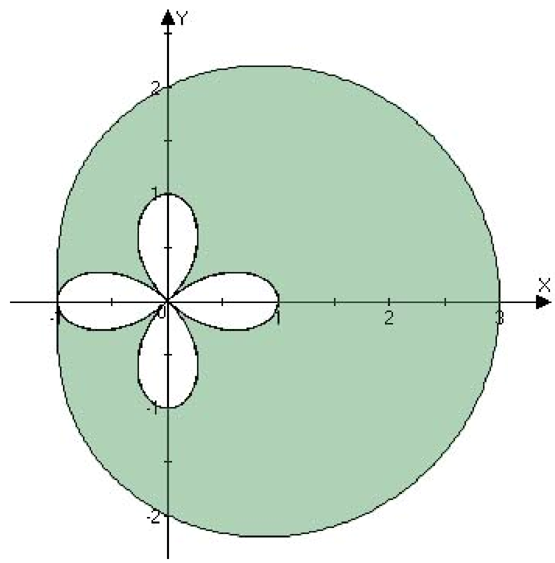
SOLN: $x = x \Leftrightarrow t_1^2 + 2 = t_2^2 + 2 \Leftarrow t_1 = -t_2$ and substituting into $y = y \Leftrightarrow t_1\left(t_1^2 - 9\right) = t_2\left(t_2^2 - 9\right)$ yields $t_1\left(t_1^2 - 9\right) = -t_1\left(t_1^2 - 9\right)$ and since $t_1 \neq t_2$, $t_1^2 - 9 = 9 - t_1^2 \Leftrightarrow t_1 = \pm 3$. So the area is

$$\int_{-3}^{3} y\frac{dx}{dt}dt = \int_{-3}^{3} t\left(t^2 - 9\right)\left(-2tdt\right) = 4\int_0^3 \left(9t^2 - t^4\right)dt$$

$$= 12t^3 - \frac{4t^5}{5}\Big|_0^3 = 81\left(4 - \frac{12}{5}\right) = \frac{648}{5} = 129.6$$

Note that this number comports well with the graph.

4. Find the area that lies inside the curve $r = 2 + \cos\theta$ and outside $r = \cos(2\theta)$.
SOLN: The curves don't cut across one another, so we can find the area inside one and outside the other and just compute the difference.
The area inside $r = 2 + \cos\theta$ is



$$2\left(\frac{1}{2}\right)\int_0^{\pi}\left(2 + \cos\theta\right)^2 d\theta = \int_0^{\pi}\left(2 + \cos\theta\right)^2 d\theta$$

$$= \int_0^{\pi} 4 + 4\cos\theta + \cos^2\theta d\theta$$

$$= 4\theta + 4\sin\theta + \frac{\theta + \sin\theta\cos\theta}{2}\Big|_0^{\pi}$$

$$= 4\pi + \frac{\pi}{2}$$

While the area inside $r = \cos(2\theta)$ is $8\left(\frac{1}{2}\right)\int_0^{\pi/4}\cos^2 2\theta d\theta = 4\left(\frac{\theta}{2} + \frac{\cos 2\theta \sin 2\theta}{4}\right)\Big|_0^{\pi/4} = \frac{\pi}{2}$

So the are (the difference in areas) is 4π.

5. The area of the surface generated by rotating the polar curve $r = f(\theta)$ for $a \le \theta \le b$ about the polar axis (the x-axis) is $S = \int_a^b 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta$. Use this formula to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

SOLN: $\frac{d}{d\theta}r^2 = 2r\frac{dr}{d\theta} = -2\sin(2\theta) \Leftrightarrow \frac{dr}{d\theta} = -\frac{\sin(2\theta)}{r} = -\frac{\sin(2\theta)}{\sqrt{\cos 2\theta}}$ so that

$$S=\int_a^b 2\pi r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta+\frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta$$

$$=4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}}\,d\theta=4\pi\int_0^{\pi/4}\sin\theta\,d\theta$$

$$=-4\pi\cos\theta\Big|_0^{\pi/4}=-4\pi\left(\frac{\sqrt{2}}{2}-1\right)=\left(2\sqrt{2}-2\right)\pi$$

6. List the first five terms of the sequence $a_1 = 2$, $a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$.

SOLN: 2,1,3,4,7,11,18,29,47,…

7. Determine whether or not the series $\sum_{n=0}^{\infty}\frac{\left(\sqrt{2}\right)^{n+1}}{2^n}$ converges. If it converges, find its sum.

SOLN: $\sum_{n=0}^{\infty}\frac{\left(\sqrt{2}\right)^{n+1}}{2^n}=\sqrt{2}\sum_{n=0}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^n=\frac{\sqrt{2}}{1-\frac{\sqrt{2}}{2}}=\frac{2\sqrt{2}}{2-\sqrt{2}}$

8. Determine whether the series $\sum_{n=1}^{\infty}\frac{2}{4n^2+32n+63}$ is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

SOLN:

$$\sum_{n=1}^{\infty}\frac{2}{4n^2+32n+63}=\sum_{n=1}^{\infty}\frac{2}{(2n+7)(2n+9)}=\sum_{n=1}^{\infty}\frac{1}{2n+7}-\frac{1}{2n+9}=\frac{1}{9}-\frac{1}{11}+\frac{1}{11}-\frac{1}{13}+\frac{1}{13}-\frac{1}{15}+\cdots=\frac{1}{9}$$

9. Find a value of c such that $\sum_{n=0}^{\infty}e^{cn}=3$

SOLN: $\sum_{n=0}^{\infty}\left(e^c\right)^n=\frac{1}{1-e^c}=3\Leftrightarrow e^c=\frac{2}{3}\Leftrightarrow \boxed{c=\ln\frac{2}{3}}$

10. Give a convincing argument as to whether the series $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ is convergent or divergent.

SOLN: It convergent since $\sum_{n=2}^{\infty}\frac{1}{n^2-1}<\sum_{n=2}^{\infty}\frac{1}{n^2-\frac{n^2}{4}}=\sum_{n=2}^{\infty}\frac{1}{\frac{3n^2}{4}}=\frac{4}{3}\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2$.

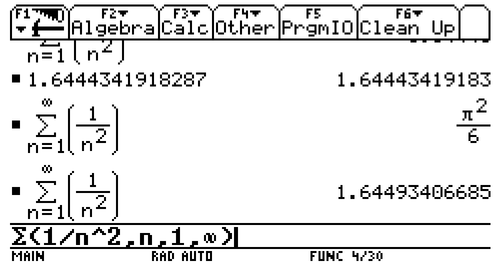
11. Use the integral inequality $s_n+\int_{n+1}^{\infty}f(x)\,dx\le s\le s_n+\int_n^{\infty}f(x)\,dx$

to estimate the number of terms needed to approximate the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ to the nearest thousandth.

SOLN: $\int_n^{\infty}\frac{1}{x^2}dx=\lim_{b\to\infty}\left(\frac{-1}{x}\right)_n^b=\lim_{b\to\infty}\left(\frac{1}{n}-\frac{1}{b}\right)=\frac{1}{n}$ so $\int_{n+1}^{\infty}\frac{1}{x^2}dx=\frac{1}{n+1}$ and $s_n+\frac{1}{n+1}\le s\le s_n+\frac{1}{n}$

We want s_n to be within half of a thousandth of s so that $\frac{1}{n} \le \frac{1}{2000} \Leftrightarrow n \ge 2000$ will do it.

To test this, we can use the TI92 (see the screen shot at right.) The first number is $\sum_{n=1}^{2000} \frac{1}{n^2} \approx 1.6444$ and the second is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$, which differs from the approximation by precisely half a thousandth. Note, however, than the rounding will be off…



12. Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converges or diverges.

SOLN: Compare with the harmonic series: $\lim_{n\to\infty} \frac{1/n}{1/(2n+3)} = \lim_{n\to\infty} \frac{2n+3}{n} = 2$, which is a constant greater than zero, so, by the limit comparison test, since the harmonic series is divergent, this one is too.

13. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1+a_n)$ is convergent.

SOLN Since $\sum a_n$ is convergent, we know that $\lim_{n\to\infty} a_n = 0$ and so the limit comparison test leads to kind of a L'Hopital's situation: $\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n}$ in that the numerator and denominator go to zero simultaneously, but a_n is not necessarily a differentiable function of n. Suppose that it is and that $a_n = f(n)$. Then L'Hospital's rule would lead to

$\lim_{n\to\infty} \frac{\ln(1+f(n))}{f(n)} = \lim_{n\to\infty} \frac{f'(n)}{(1+f(n))f'(n)} = \lim_{n\to\infty} \frac{1}{1+f(n)} = 1$, which means the series both converge.

What if there is no such function? Then try comparison. By definition of the limit, for any $\varepsilon > 0$ there exists M such that if $n > M$ then $a_n < \varepsilon$. We can choose $\varepsilon = 1$ so that if $n > M$ then $a_n < 1$ which means that $0 < \log(1 + a_n) < a_n$. So the series converges by direct comparison.

14. Test the series $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{(n-2)\pi}{2n}\right)$ for convergence or divergence.

SOLN: $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{(n-2)\pi}{2n}\right) = 0 + 1 - 0 + \frac{\sqrt{2}}{2} - \cos\left(\frac{3\pi}{10}\right) + \cdots$ converges by the alternating series test: $\lim_{n\to\infty} \cos\left(\frac{(n-2)\pi}{2n}\right) = \cos\left(\lim_{n\to\infty} \frac{(n-2)\pi}{2n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi - 2\pi/n}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

15. How many terms are needed to approximate the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ to the nearest ten thousandth?

SOLN: The series is alternating, so the error in approximation is less than the magnitude of first neglected term. The smallest value of n so that $1/n!$ is less than a half of a ten thousandths is $n = 8$.