Math 1B—Calculus II Fair Game for Test 4

1. By completing the squares for x and y, it is easy to see that the equation $x^2 + 2x + y^2 + 6y = 0$ describes an ellipse with a center at (–1, –3).
   a. Write the equation for the ellipse in standard form by specifying values for a, b, h, and k in the formula $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b}=1$
   b. Find parametric equations for this ellipse. Recall the identity $\cos^2 t+\sin^2 t=1$
   c. Find an equation for the line tangent to the ellipse at the origin: $(x,y) = (0,0)$.
   d. Sketch a graph for the ellipse and the tangent line together.

2. By completing the squares for x and y, it is easy to see that the equation $x^2 - 6x - y^2 + 8y = 0$ describes a hyperbola with a center at (–3, –4).
   a. Write the equation for the ellipse in standard form by specifying values for a, b, h, and k in the formula $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b}=1$
   b. Find parametric equations for this hyperbola. Recall the identity $\sec^2 t-\tan^2 t=1$
   c. Find an equation for the line tangent to the hyperbola at the origin: $(x,y) = (0,0)$.
   d. Sketch a graph for the hyperbola and the tangent line together.

3. Consider the polar function $r = 4 + \sin(4\theta)$. Find the area enclosed by this curve.

4. The parametric equations $\boxed{\begin{array}{l} x=|t| \\ y=\sin t-2\sin^3 t \end{array}}$ describe a curve which forms a sequence of loops around the x-axis in the xy-plane.
   Find the area of the loop closest to the origin.

5. Find the area of the region outside the polar curve $r = 2$ and inside the polar curve $r = 4\cos\theta$.

6. Find the length of the curve described by the parametric equations,
   $$x = 1 + 2\cos^3(t) \quad \text{and} \quad y = 2 - 3\sin^2(t)$$

7. A curve is defined by the parametric equations,
   $$x=\int_0^t \sin\left(u^2\right)du \qquad y=\int_0^t \cos\left(u^2\right)du$$
   Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

8. Find a value of r so that $\sum_{n=0}^{\infty} r^n = 17$.

9. Use the limit comparison test to determine whether $\sum_{n=10}^{\infty}\dfrac{4n^2+13n+3}{3n^3+8n^2+5n}$ is convergent or not.

10. Find the value of the sum $\sum_{n=1}^{\infty} \frac{3}{9n^2+3n-2}$

11. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n 100^n}{(2n)!}$.

a. **Explain** why the sequence $\left\{\frac{100^n}{(2n)!}\right\}_{n=0}^{\infty}$ converges to zero, so that the series $\sum_{n=0}^{\infty} \frac{(-1)^n 100^n}{(2n)!}$ satisfies the *n*th term test.

b. **Explain** why the series $\sum_{n=0}^{\infty} \frac{(-1)^n 100^n}{(2n)!}$ is convergent and use the alternating series error bound to find an N so that $\sum_{n=0}^{N} \frac{(-1)^n 100^n}{(2n)!}$ approximates $\sum_{n=0}^{\infty} \frac{(-1)^n 100^n}{(2n)!}$ to within one billionth (10^{-9}).

12. Show that the series $\sum_{n=100}^{\infty} \frac{1}{n^{1.01}+1}$ is convergent by using a comparison test and the integral test.

13. Determine whether each series converges or diverges.

a. Using the ratio test, $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$ converges: $\lim_{n\to\infty} \frac{(n+1)^{10}}{2^{n+1}} \frac{2^n}{n^{10}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{10} \frac{1}{2} = \frac{1}{2} < 1$

b. Writing out the first few terms, $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n}{2}\right)}{n} = 1+0-\frac{1}{3}-0+\frac{1}{5}+0-\frac{1}{7}+0+\frac{1}{9}-\cdots$

Evidently, $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n}{2}\right)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$ (The last equality is familiar from discussion in class.) This series is converging by the alternating series test.

c. $\sum_{n=1}^{\infty} \frac{(n-1)!}{10^n}$ This series fails the *n*th term test, but let's use the ratio test to show it's divergent: $\lim_{n\to\infty} \frac{n!}{10^{n+1}} \frac{10^n}{(n-1)!} = \lim_{n\to\infty} \frac{n}{10} = \infty > 1$

d. $\sum_{n=1}^{\infty} \frac{n^{2.01}-n}{n^3+2}$ This series is divergent. Use limit comparison with the divergent *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$: $\lim_{x\to\infty} \frac{n^{2.01}-n}{n^3+2} \frac{n^{0.99}}{1} = \lim_{x\to\infty} \frac{n^3-n^{1.99}}{n^3+2} = 1$.

3. The error in approximating $S=\sum_{n=1}^{\infty}\frac{1}{n^2}$ by $S_{99}=\sum_{n=1}^{99}\frac{1}{n^2}$ is $\left|S-S_{99}\right|$ where

$$0.01=\lim_{b\to\infty}\left.\frac{-1}{x}\right|_{100}^{b}=\int_{100}^{\infty}\frac{dx}{x^2}<\left|S-S_{99}\right|<\int_{99}^{\infty}\frac{dx}{x^2}=\lim_{b\to\infty}\left.\frac{-1}{x}\right|_{99}^{b}=\frac{1}{99}\approx 0.0101010101$$

4. Since $S=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}$ is an alternating series and passes the alternating series test for convergence, The the error in approximating $S\approx S_N$ is less than the first neglected term:

$\left|S-S_N\right|=\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}-\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)!}\right|<\frac{1}{(2(N+1)+1)!}$. So, choose N so that $(2N+3)!\geq 2\times 10^{12}\Rightarrow N\geq 6$, In fact, from MacLaurin series for sine we know this sums to $\sin(1)\approx 0.841470984808$. Summing to $N = 5$ is not quite good enough:

$$\sum_{n=0}^{5}\frac{(-1)^n}{(2n+1)!}=1-\frac{1}{6}+\frac{1}{120}-\frac{1}{5040}+\frac{1}{39916800}-\frac{1}{65570520800}\approx 0.841470984809$$

5. The radius of convergence of $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^{2n}$ is found using the ratio test:

$$\lim_{n\to\infty}\frac{(2n+2)!x^{2n+2}}{((n+1)!)^2}\frac{(n!)^2}{(2n)!x^{2n}}=\lim_{n\to\infty}\frac{(2n+2)(2n+1)x^2}{(n+1)^2}=4x^2<1\Leftrightarrow x\in\left(-\frac{1}{2},\frac{1}{2}\right)$$

So the radius of convergence is $R = ½$. What happens at the endpoints?

6. $$\int_0^{0.2}\frac{dx}{2+x^3}=\frac{1}{2}\int_0^{0.2}\frac{dx}{1-\left(-\frac{x^3}{2}\right)}=\frac{1}{2}\int_0^{0.2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{2^n}dx=\sum_{n=0}^{\infty}\left.\frac{(-1)^n x^{3n+1}}{2^{n+1}(3n+1)}\right|_0^{0.2}$$

$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}5^{3n+1}(3n+1)}\approx\frac{1}{10}-\frac{1}{10000}=0.0999$, which is accurate to within 0.0001 since the next term is a bound on the error and is much smaller than 0.0001