Math 1B –Test 5 – Fair Game

1. Consider the function $f(x)=\cos(2x)\cos(3x)$
   a. Use the trig identity $\cos(\alpha)\cos(\beta)=\frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)$ to show that $f(x)=\frac{1}{2}\cos(x)+\frac{1}{2}\cos(5x)$ and combine Maclaurin series for $\cos x$ and $\cos 5x$ to find $T_4(x)$, the fourth degree Maclaurin polynomial for $f(x)$.
   b. Write Maclaurin polynomials for $\cos(2x)$ and $\cos(3x)$ multiply these to verify your result in part (a).

2. Let $g(x)=x^2e^{-x}$
   a. Write a formula for the Maclaurin coefficient of *g*.
   b. Use the Maclaurin series to approximate $g\left(\frac{1}{2}\right)$ to the nearest thousandth.
   c. What are the fewest number of terms needed for the approximation in part (b)?

3. Find the first three non-zero terms of the Maclaurin series for $f(x)=\int_0^x \arctan t\,dt$.

4. Consider $r(x)=2\left(1-\frac{x}{16}\right)^{1/4}$ using a binomial series.
   a. Find the first 4 terms in the Maclaurin series for $r(x)$.
   b. How many terms are necessary to approximate its value to within 5×10^{-13}?
   c. What is the radius of convergence for $r(x)$?

5. Use the Maclaurin series for e^x to develop a formula for the Maclaurin coefficients of $e^{-x^2/2}$.

6. Use the Maclaurin series for $\cos x$ and $\sin x$ to find the first three Maclaurin coeffients of $\cos(2x)\sin(3x)$.

7. Find the three non-zero terms of the Taylor polynomial for $f(x)=\int_1^x \ln\left(1+t^2\right)dt$ about $a=1$.

8. Approximate $(1.1)^{1/3}$ using a binomial series. How many terms are necessary to approximate its value to within 0.00001 ?

9. Find the 6th degree Maclaurin polynomial for $f(x)=\left(1-x^2\right)^{-1/4}$ and use it to approximate $\int_0^{0.1}\frac{dx}{\left(1-x^2\right)^{1/4}}$.

   What does the Taylor inequality tell you about the error in approximation?

Answers: (DON'T LOOK UNTIL YOU'RE DESPERATE OR JUST WANT TO CONFIRM☺

1. (a) ANS: $f(x) = \cos(2x)\cos(3x) = \frac{1}{2}\cos(2x-3x) + \frac{1}{2}\cos(2x+3x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(5x)$

   Thus, $T_4(x) = \frac{1}{2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + 1 - \frac{(5x)^2}{2} + \frac{(5x)^4}{24}\right) = 1 - \frac{13x^2}{2} + \frac{313x^4}{24}$

   (b) ANS: $f(x) = \cos(2x)\cos(3x) \approx \left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24}\right)\left(1 - \frac{(3x)^2}{2} + \frac{(3x)^4}{24}\right)$

   $\approx 1 - \frac{(9+4)x^2}{2} + \left(\frac{81}{24} + \frac{36}{4} + \frac{16}{24}\right)x^4 = 1 - \frac{13x^2}{2} + \frac{313}{24}x^4$

2. (a) ANS: $g(x) = x^2 e^{-x} - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} x^n$, so the coefficient of x^n is 0 if $n < 2$ and for $n > 1$, $c_n = \frac{(-1)^n}{(n-2)!}$

   (b) $g\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!}\left(\frac{1}{2}\right)^n = \frac{1}{4} - \frac{1}{8} + \frac{1}{32} - \frac{1}{192} + \frac{1}{1536} - \cdots$

   (c) Evidently we need at least these five terms to obtain $g\left(\frac{1}{2}\right) \approx 0.152$

   $$\sum_{n=2}^{5} \frac{(-1)^n}{(n-2)!2^n} = .15104$$

   $$\sum_{n=2}^{6} \frac{(-1)^n}{(n-2)!2^n} = .15170$$

   $$\sum_{n=2}^{7} \frac{(-1)^n}{(n-2)!2^n} = .15163$$

3. ANS: $f(x) = \int_0^x \arctan t\,dt \Rightarrow f''(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$, Integrating twice yields

   $f(x) = c_0 + c_1 x + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30}$. Here $c_0 = f(0) = 0$ and $c_1 = \arctan(0) = 0$

4. (a) ANS:

   $$r(x) = 2\left(1 - \frac{x}{16}\right)^{1/4} = 2\sum_{n=0}^{\infty}\binom{1/4}{n}\left(\frac{-x}{16}\right)^n \approx 2\left(1 + \frac{1}{4}\left(\frac{-x}{16}\right) + \frac{\frac{1}{4}\left(\frac{-3}{4}\right)}{2!}\left(\frac{-x}{16}\right)^2 + \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)}{3!}\left(\frac{-x}{16}\right)^3\right)$$

   $= 2 - \frac{x}{32} - \frac{3x^2}{2048} - \frac{7x^3}{262144}$

   (b) ANS: This, of course, will depend on the value of *x*. If *x* is negative then the series is alternating and it is sufficient to require that the number of terms *N* is such that

$$2\binom{1/4}{N+1}\left(\frac{x}{16}\right)^{N+1} < 5\times10^{-13} \Leftrightarrow \log 2+\log\binom{1/4}{N+1}+(N+1)\log\left(\frac{x}{16}\right) < -13+\log(5)$$

(c) The radius of convergence is found by requiring that

$$\lim_{n\to0}\left|\frac{a_{n+1}}{a_n}\right|<1 \Leftrightarrow \lim_{n\to0}\frac{1/4(-3/4)\cdots(1/4-n+1)(1/4-n)n!}{1/4(-3/4)\cdots(1/4-n+1)(n+1)!}\left|\frac{x^{n+1}}{16^{n+1}}\right|\left|\frac{16^n}{x^n}\right| = \lim_{n\to0}\left|\frac{1/4-n}{16(n+1)}\right| = \frac{|x|}{16}<1,$$ so the radius of convergence is 16.

5. $e^{-x^2/2} = \sum_{n=0}^{\infty}\frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}$ so the Maclaurin coefficients are $c_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{(-1)^n}{2^n n!} & \text{if } n \text{ is even}\end{cases}$

6. $\cos(2x)\sin(3x) = \sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{(2n+1)!} \approx \left(1-2x^2+\frac{2}{3}x^4\right)\left(3x-\frac{9x^3}{2}+\frac{81x^5}{40}\right)$

$\approx 3x-\left(6+\frac{9}{2}\right)x^3+\left(\frac{81}{40}+9+2\right)x^5 = 3x-\frac{21}{2}x^3+\frac{521}{40}x^5$

7. $c_0 = f(1) = \int_1^1 \ln(1+t^2)\,dt \equiv 0$,

$c_1 = f'(1) = \ln(2)$

$c_2 = \frac{f''(1)}{2} = \frac{1}{2}\frac{d}{dt}\ln(1+t^2)\Big|_{t=1} = \frac{t}{1+t^2}\Big|_{t=1} = \frac{1}{2}$

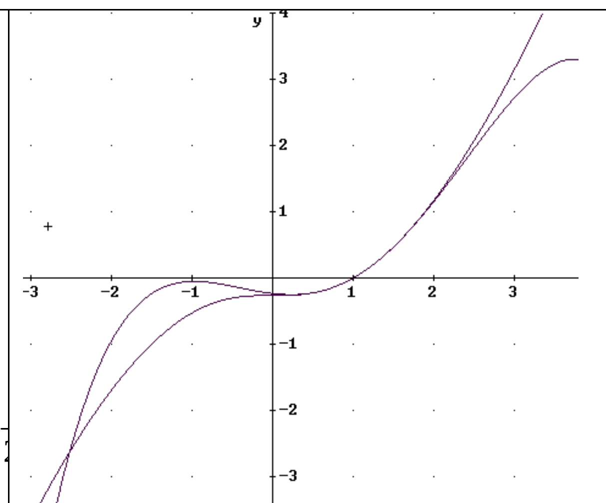
$c_3 = \frac{f''(1)}{6} = \frac{1}{3}\frac{d}{dt}\frac{t}{1+t^2}\Big|_{t=1} = \frac{1}{3}\frac{1-t^2}{(1+t^2)^2}\Big|_{t=1} = 0$

$c_4 = \frac{f^{(4)}(1)}{24} = \frac{1}{24}\frac{d}{dt}\frac{2(1-t^2)}{(1+t^2)^2}\Big|_{t=1} = \frac{1}{6}\frac{t(t^2-3)}{(1+t^2)^3}\Big|_{t=1} = -$

So, near $x = 1$,

$f(x) = \ln 2(x-1)+\frac{(x-1)^2}{2}-\frac{(x-1)^4}{24}$

See the graph at right:



8. Let $f(x) = (1+x)^{1/3} = \sum_{n=0}^{\infty}\binom{1/3}{n}x^n = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\frac{22}{729}x^5-\cdots$

Then $(1.1)^{1/3} = f(0.1) = \sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{1}{10}\right)^n = 1+\frac{1}{30}-\frac{1}{900}+\frac{5}{81000}-\frac{1}{243000}+\cdots$

Since the series is alternating the first five terms shown here are sufficient to approximate within 0.00001: $1+\frac{1}{30}-\frac{1}{900}+\frac{5}{81000}-\frac{1}{243000} = 1.03227983539 \approx 1.032280115$

9. $f\left(x\right)=\left(1-x^2\right)^{-1/4}=\sum_{n=0}^{\infty}\binom{-1/4}{n}\left(-1\right)^n x^{2n}=1+\frac{1}{4}x^2+\frac{5}{32}x^4+\frac{15}{128}x^6+\cdots$

$$\int_0^{0.1}\frac{dx}{\left(1-x^2\right)^{1/4}}\approx\int_0^{0.1}1+\frac{1}{4}x^2+\frac{5}{32}x^4+\frac{15}{128}x^6dx=x+\frac{x^3}{12}+\frac{x^5}{32}+\frac{15x^7}{896}\Bigg|_0^{0.1}$$

$$=\frac{1}{10}+\frac{1}{12000}+\frac{1}{3200000}+\frac{3}{1732000000}=0.1000836475\underline{\mathbf{0}}74$$

If $g\left(x\right)=\int_0^x\frac{dt}{\left(1-t^2\right)^{1/4}}$, then, using a calculator, we can find the maximum value of $g^{(8)}\left(x\right)\le 435.8$, so Taylor's inequality will have the above integral approximation accurate to withing $\frac{435.8\left(0.1\right)^8}{8!}\approx 1.0808\times 10^{-10}$. Indeed, using a calculator, we have $g\left(0.1\right)\approx 0.1000836475\underline{\mathbf{1}}81$