# Archimedes' Principle and Problems Plus #6 From Chapter 6 of Stewart's Early Transcendentals Calculus

Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density $\rho_o$ floating partly submerged in a fluid of density $\rho_f$, the buoyant force is given by

$F = \rho_f g \int_{-h}^{0} A(y)\,dy$, where $g$ is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object. The weight of the object is given by

$$W = \rho_o g \int_{-h}^{L-h} A(y)\,dy$$

(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100 \frac{\rho_f - \rho_o}{\rho_f}$$

SOLN: The portion of the of the volume of the object above the surface is

$$\frac{\text{volume above}}{\text{total volume}} = \frac{\int_{0}^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \frac{\rho_f g \int_{-h}^{L-h} A(y)\,dy - \rho_f g \int_{-h}^{0} A(y)\,dy}{\rho_f g \int_{-h}^{L-h} A(y)\,dy}$$

Now

$\rho_f g \int_{-h}^{0} A(y)\,dy = \rho_o g \int_{-h}^{L-h} A(y)\,dy$ is the weight of the object (it displaces its weight in the fluid) and

the result is then apparent.

(b) The density of ice is 917 kg/m$^3$ and the density of seawater is 1030 kg/m$^3$ What percentage of the volume of an iceberg is above water?

SOLN: This is just a computation: $\dfrac{1030 - 917}{1030} = \dfrac{113}{1030} \approx 11\%$

(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?

SOLN: No – the ice displaces its weight in the water. When it melts its mass density increases so that its liquid form exactly fits the volume it displaced in the cup. Try it!

(d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is 1000 Kg/m$^3$.

SOLN: This is the same work required to lift a sphere of that radius entirely submerged the length of its diameter against the gravitational field. The weight of the sphere is $1000(4/3)\pi(0.4)^3 = 256\pi/3$Kg. To get the force density, multiply this quantity by the acceleration due to gravity. Finally this entire weight is to be lifter one diameter, so the work done is $0.8(9.81)256\pi/3$ Joules, or about 2100 J.