Some interesting integrals from Stewart (Early Transcendentals §7.4

7.4 #24: We seek an antiderivative for $\int \frac{x^2+x+1}{\left(x^2+1\right)^2}dx$

Observe that the denominator is the square of an irreducible quadratic and so the partial fractions expanded form for this integrand is $\frac{x^2+x+1}{\left(x^2+1\right)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2} \Leftrightarrow \left(Ax+B\right)\left(x^2+1\right)+Cx+D=x^2+x+1$

Expanding the LHS and equating coefficients, we have

$$Ax^3+Bx^2+\left(A+C\right)x+B+D=x^2+x+1 \Leftrightarrow A=0, B=1, C=1, D=0$$

So, $\int \frac{x^2+x+1}{\left(x^2+1\right)^2}dx=\int \frac{1}{x^2+1}+\frac{x}{\left(x^2+1\right)^2}dx=\arctan\left(x\right)+\int \frac{x}{\left(x^2+1\right)^2}dx$

$\int \frac{x}{\left(x^2+1\right)^2}dx=\frac{1}{2}\int \frac{2x}{\left(x^2+1\right)^2}dx=\frac{1}{2}\int \frac{du}{u^2}=-\frac{1}{2\left(x^2+1\right)}$ so $\boxed{\int \frac{x^2+x+1}{\left(x^2+1\right)^2}dx=\arctan\left(x\right)-\frac{1}{2\left(x^2+1\right)}+c}$

7.4 #30: We seek an antiderivative for $\int \frac{3x^2+x+4}{x^4+3x^2+2}dx$.

Observe that the denominator factors as a product of irreducible quadratics, so the partial fractions expansion is

$$\frac{3x^2+x+4}{x^4+3x^2+2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2} \Leftrightarrow 3x^2+x+4=\left(Ax+B\right)\left(x^2+2\right)+\left(Cx+D\right)\left(x^2+1\right)$$

Expand and equate coefficients: $3x^2+x+4=\left(A+C\right)x^3+\left(B+D\right)x^2+\left(2A+C\right)x+2B+D$

$A+C=0;\ B+D=3;\ 2A+C=1;\ \ 2B+D=4$. This works out to two 2X2 linear systems which are easily solved: $A=1,\ B=1,\ C=-1,\ \ D=2$ so that

$$\begin{aligned}
\int \frac{3x^2+x+4}{x^4+3x^2+2}dx &= \int \frac{x+1}{x^2+1}-\frac{x-2}{x^2+2}dx=\int \frac{x}{x^2+1}+\frac{1}{x^2+1}-\frac{x}{x^2+2}+\frac{2}{x^2+2}dx \\
&= \frac{1}{2}\int \frac{2x}{x^2+1}dx+\int \frac{1}{x^2+1}dx-\frac{1}{2}\int \frac{2x}{x^2+2}dx+\int \frac{2}{x^2+2}dx \\
&= \frac{1}{2}\ln\left(x^2+1\right)+\arctan\left(x\right)-\frac{1}{2}\ln\left(x^2+2\right)+\int \frac{2}{2u^2+2}\sqrt{2}du \\
&= \frac{1}{2}\ln\frac{x^2+1}{x^2+2}+\arctan\left(x\right)+\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)+c
\end{aligned}$$

#38 We seek an antiderivative for $\int \frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2}dx$

Observe that the denominator is the square of an irreducible quadratic (discriminant is 4 – 8)

To simplify matters, complete the square in the denominator: $x^2+2x+2=\left(x+1\right)^2+1$ and substitute $u = x + 1$ (so that $x = u - 1$.)

$$\begin{aligned}\int \frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2}dx &= \int \frac{\left(u-1\right)^3+2\left(u-1\right)^2+3\left(u-1\right)-2}{\left(u^2+1\right)^2}dx \\ &= \int \frac{u^3-3u^2+3u-1+2u^2-4u+2+3u-3-2}{\left(u^2+1\right)^2}du \\ &= \int \frac{u^3-u^2+2u-4}{\left(u^2+1\right)^2}du \\ &= \int \frac{Au+B}{u^2+1}+\frac{Cu+D}{\left(u^2+1\right)^2}du\end{aligned}$$

So we find A,B,C,D by equating coefficients in

$$\begin{aligned}u^3-u^2+2u-4 &= \left(Au+B\right)\left(u^2+1\right)+\left(Cu+D\right) \\ &= Au^3+Bu^2+\left(A+C\right)u+B+D\end{aligned}$$

Whence $A = 1$, $B = -1$, $C = 1$, $D = -3$ so

$$\begin{aligned}\int \frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2}dx &= \int \frac{u-1}{u^2+1}+\frac{u-3}{\left(u^2+1\right)^2}du \\ &= \frac{1}{2}\int \frac{2u}{u^2+1}du-\int \frac{1}{u^2+1}du+\frac{1}{2}\int \frac{2u}{\left(u^2+1\right)^2}du-\int \frac{3}{\left(u^2+1\right)^2}du \\ &= \frac{1}{2}\ln\left(u^2+1\right)-\arctan\left(u\right)-\frac{1}{2\left(u^2+1\right)}-\int \frac{3\sec^2\theta}{\left(\tan^2\theta+1\right)^2}d\theta\end{aligned}$$

The last integral is using the substitution, $u=\tan\theta$ so that $du=\sec^2\theta d\theta$. Thus we get

$$3\int \cos^2\theta d\theta = 3\int \frac{1+\cos 2\theta}{2}d\theta = \frac{3\theta}{2}+\frac{3\sin\theta\cos\theta}{2}+c=\frac{3}{2}\arctan\left(x+1\right)+\frac{3\left(x+1\right)}{2\left(x^2+2x+2\right)}+c$$

Putting it all together,

$$\boxed{\int \frac{x^3+2x^2+3x-2}{\left(x^2+2x+2\right)^2}dx = \frac{1}{2}\ln\left(x^2+2x+2\right)+\frac{1}{2}\arctan\left(x+1\right)-\frac{1}{2\left(x^2+2x+2\right)}+\frac{3\left(x+1\right)}{2\left(x^2+2x+2\right)}+c}$$