Math 1B Take-home problems for Test 3 – spring '10

Consider the following three theorems (note ξ is the Greek letter "xi" which is pronounced "zai" – rhyming with "sigh"):

Theorem 1: If f is a continuously differentiable function on $[a, b]$ for which $f''(u)$ exists at each point u of (a, b) and if T_n is the n-subdivision trapezoidal rule approximation to $\int_a^b f(t)\,dt$, then there exists ξ in (a, b) such that

$$\int_a^b f(t)\,dt = T_n - f''(\xi)\frac{(b-a)^3}{12n^2}$$

Theorem 2: If f is a continuously differentiable function on $[a, b]$ for which $f''(u)$ exists at each point u of (a, b) and if M_n is the n-subdivision midpoint rule approximation to $\int_a^b f(t)\,dt$, then there exists ξ in (a, b) such that

$$\int_a^b f(t)\,dt = M_n + f''(\xi)\frac{(b-a)^3}{24n^2}$$

Theorem 3: If f is a continuously differentiable function on $[a, b]$ for which $f^{(4)}(u)$ exists at each point u of (a, b) and if S_{2n} is the $2n$-subdivision Simpson rule approximation to $\int_a^b f(t)\,dt$, then there exists ξ in (a, b) such that

$$\int_a^b f(t)\,dt = S_{2n} - f^{(4)}(\xi)\frac{(b-a)^5}{180(2n)^4}$$

These theorems are "existence" theorems in that they don't say how to find zeta, just that such a ξ exists. Give detailed explanations as to how to find ξ for all three theorems for each of the following integrals and values of n. Use a computing device only sparingly, as needed:

1. $\int_0^1 x^4\,dx$ for $n = 2$.
2. $\int_0^1 x^5\,dx$ for $n = 2$.
3. $\int_0^1 x^4\,dx$ for $n = 4$.
4. $\int_0^1 x^5\,dx$ for $n = 4$.
5. $\int_0^1 \frac{1}{x^2+1}\,dx$ for $n = 4$.
6. $\int_0^{\pi/2} \sin^4(x)\,dx$ for $n = 4$.

Math 1B Take-home problems Solutions for Test 3 – spring '10

1. $\int_0^1 x^4 dx$ for $n = 2$.

$$T_2 = \frac{f(0)+2f(1/2)+f(1)}{2}\left(\frac{1-0}{2}\right) = \frac{9}{32} \text{ and } f''(\xi) = 12\xi^2, \text{ so}$$

$$\int_0^1 f(x)\,dx = T_2 - f''(\xi)\frac{(1-0)^3}{12(2)^2} \text{ is the equation,}$$

$$\frac{1}{5} = \frac{9}{32} - 12\xi^2\frac{1}{48} \Leftrightarrow \frac{\xi^2}{4} = \frac{13}{160} \Leftarrow \boxed{\xi = \sqrt{\frac{13}{40}} = \frac{\sqrt{130}}{20} \approx 0.57008771255}$$

$$M_2 = \left(f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right)\left(\frac{1-0}{2}\right) = \frac{82}{256}\left(\frac{1}{2}\right) = \frac{41}{256}, \text{ so the midpoints equation is}$$

$$\frac{1}{5} = \frac{41}{256} + 12\xi^2\frac{(1-0)^3}{24(2)^2} \Leftrightarrow \frac{\xi^2}{8} = \frac{1}{5} - \frac{41}{256} \Leftrightarrow \xi^2 = \frac{256-205}{160} \Leftarrow \boxed{\xi = \frac{\sqrt{510}}{40} \approx 0.564579489532}$$

$$S_4 = \frac{T_2 + 2M_2}{3} = \frac{\frac{9}{32}+\frac{41}{128}}{3} = \frac{77}{384}, \text{ so that } \frac{1}{5} = \frac{77}{384} - 24\frac{(1-0)^5}{180(4)^4} = \frac{77}{384} - \frac{2}{3840} = \frac{768}{3840} = \frac{1}{5} \text{ is}$$

true…for all ξ in (0,1).

2. $\int_0^1 x^5 dx$ **for $n = 2$.**

$$T_2 = \frac{f(0)+2f(1/2)+f(1)}{2}\left(\frac{1-0}{2}\right) = \frac{17}{64} \textbf{ and } f''(\xi) = 20\xi^3\textbf{, so}$$

$$\int_0^1 x^5 dx = T_2 - f''(\xi)\frac{(1-0)^3}{12(2)^2} \textbf{ is the equation,}$$

$$\frac{1}{6} = \frac{17}{64} - 20\xi^3\frac{1}{48} \Leftrightarrow \frac{5\xi^3}{12} = \frac{19}{192} \Leftarrow \boxed{\xi = \sqrt[3]{\frac{19}{80}} = \frac{\sqrt[3]{1900}}{20} \approx 0.619281164815}$$

$$M_2 = \left(f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right)\left(\frac{1-0}{2}\right) = \frac{61}{512}\textbf{, so the midpoints equation is}$$

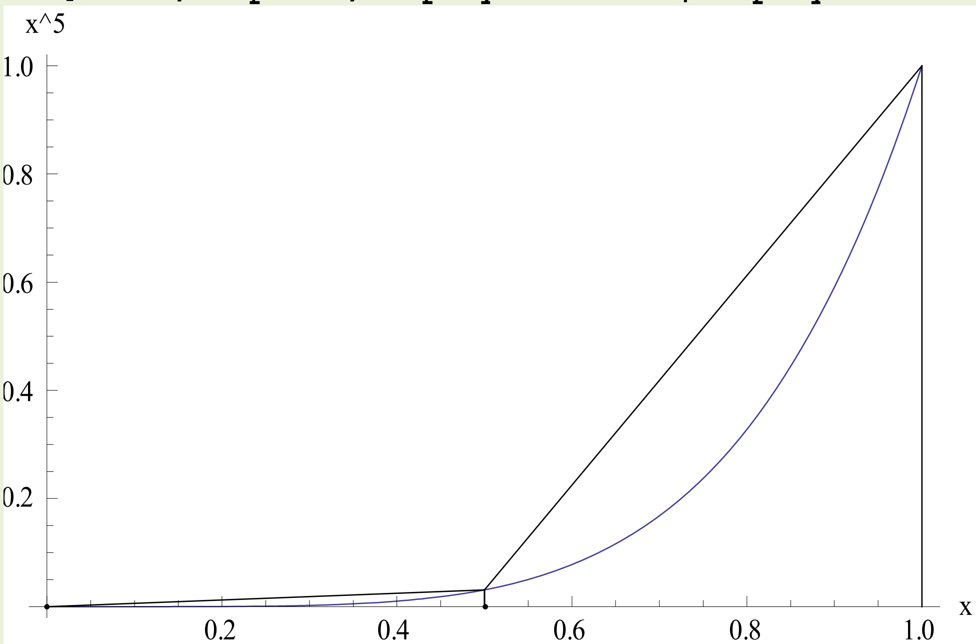
$$\frac{1}{6} = \frac{61}{512} + \frac{5\xi^3}{24} \Leftrightarrow \frac{5\xi^3}{24} = \frac{1}{6} - \frac{61}{512} \Leftrightarrow \xi^3 = \frac{24}{5}\left(\frac{73}{1536}\right) = \frac{73}{320} \Leftarrow \boxed{\xi = \frac{\sqrt[3]{1825}}{20} \approx 0.611023097211}$$

$$S_4 = \frac{T_2 + 2M_2}{3} = \frac{\frac{17}{64}+\frac{61}{256}}{3} = \frac{43}{256}\textbf{, so that } \frac{1}{6} = \frac{43}{256} - 120\xi\frac{(1-0)^5}{180(4)^4} = \frac{43}{256} - \frac{\xi}{384} \Leftrightarrow \boxed{\xi = \frac{1}{2}}\textbf{.}$$

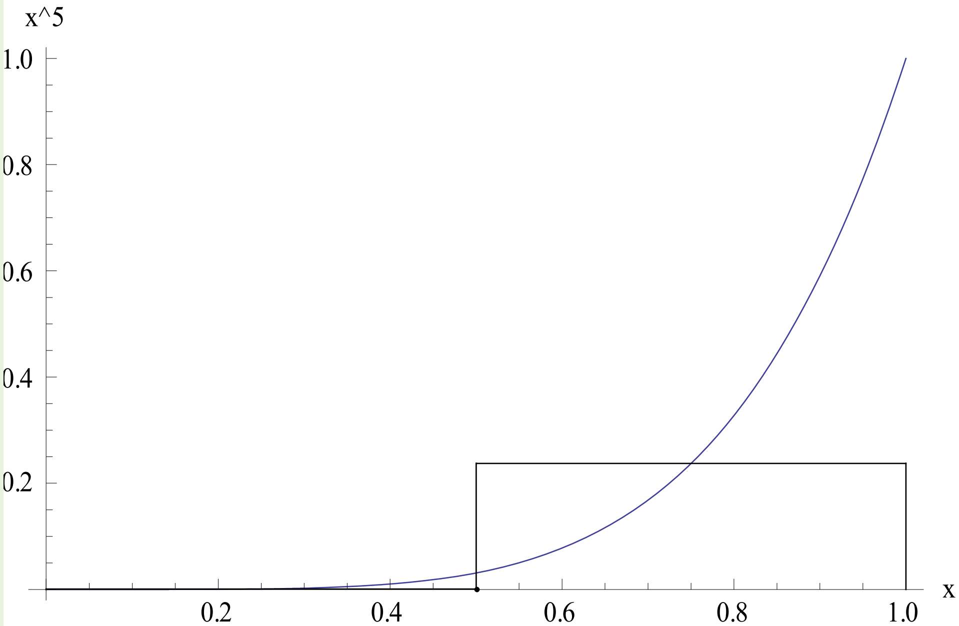
Here's some Mathematica Notebook commands I wrote to verify these:

```
f[x_]:=x^5;
```

```
fPlot =
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Ident
ity];
a=0;b=1;n=2;h=(b-a)/n;points =
Partition[Flatten[Table[{{i,a},{i,f[i]},{i+b/n,f[i +
b/n]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
trapPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,trapPlot,DisplayFunction→$DisplayFunction]
```




```
trapsum = 0; Do[ trapsum = trapsum +2*f[a+i*h],{i,1,n-1}];
trapsum = trapsum + f[a]+f[b];
trapsum = h/2*trapsum; Print["The trapezoidal sum for the
integral is ", trapsum]
 The trapezoidal sum for the integral is  17/64
Solve[Integrate[f[x],{x,0,1}]==trapsum-f''[xi]*(b-
a)^3/(12*n^2),xi]
 {{xi→-(1/2) (-(19/10))^(1/3)},{xi→1/2 (19/10)^(1/3)},{xi→1/2 (-1)^(2/3)
(19/10)^(1/3)}}
N[%]
 {{xi→-0.309641-0.536313 i},{xi→0.619281},{xi→-
0.309641+0.536313 i}}
fPlot =
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Ident
ity];
a=0;b=1;n=2;h=(b-a)/n;points = Partition[
Flatten[Table[{{i,a},{i,f[i+b/(2*n)]},{i+b/n,f[i +
b/(2*n)]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
midptPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,midptPlot,DisplayFunction→$DisplayFunction]
```




```
midptsum = 0; Do[ midptsum = midptsum +f[a+(2*i-
1)*h/2],{i,1,n}]; midptsum = h*midptsum; Print["The midpoint
sum for the integral is ", midptsum]
The midpoint sum for the integral is  61/512
Solve[Integrate[f[x],{x,0,1}]==midptsum+f''[xi]*(b-
a)^3/(24*n^2),xi]
{{xi→-(1/4) (-(73/5))^(1/3)},{xi→1/4 (73/5)^(1/3)},{xi→1/4 (-1)^(2/3)
(73/5)^(1/3)}}
N[%]
{{xi→-0.305512-0.529162 i},{xi→0.611023},{xi→-
0.305512+0.529162 i}}
simpsonSum = (trapsum + 2*midptsum)/3;Print["The midpoint
sum for the integral is ", simpsonSum]
The midpoint sum for the integral is  43/256
Solve[Integrate[f[x],{x,0,1}]==simpsonSum-f''''[xi]*(b-
a)^5/(180*(2*n)^4),xi]
{{xi→1/2}}
```

3. $\int_0^1 x^4 dx$ **for $n = 4$.**

$T_4 = \dfrac{f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)}{2}\left(\dfrac{1-0}{4}\right) = \dfrac{113}{512}$ **and** $f''(\xi)=12\xi^2$**, so**

$\int_0^1 x^4 dx = T_4 - f''(\xi)\dfrac{(1-0)^3}{12(4)^2}$ **is the equation,**

$$\frac{1}{5}=\frac{113}{512}-\xi^2\frac{1}{16} \Leftrightarrow \frac{\xi^2}{16}=\frac{53}{2560} \Leftarrow \boxed{\xi=\sqrt{\frac{530}{1600}}=\frac{\sqrt{530}}{40}\approx 0.575543221661}$$

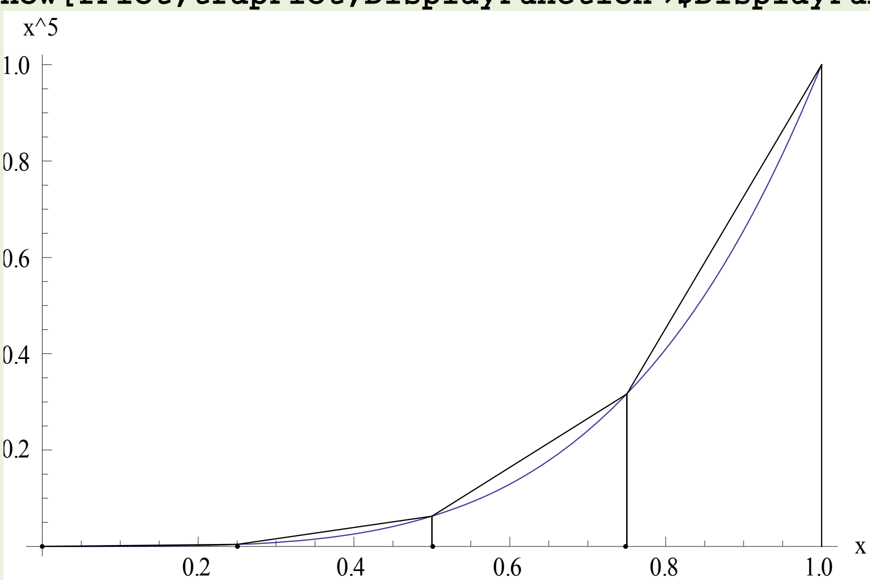
$M_4 = \left(f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right)\left(\frac{1-0}{4}\right)=\frac{777}{4096}$**, so the midpoints equation is**

$$\frac{1}{5}=\frac{777}{4096}+\frac{\xi^2}{32}\Leftrightarrow\frac{\xi^2}{32}=\frac{1}{5}-\frac{777}{4096}\Leftrightarrow\xi^2=\frac{211}{640}\Leftarrow\boxed{\xi=\frac{\sqrt{2110}}{80}\approx 0.574184203893}$$

$$S_8=\frac{T_4+2M_4}{3}=\frac{\frac{113}{512}+\frac{777}{2048}}{3}=\frac{1229}{6144}$$, **so that** $$\frac{1}{5}=\frac{1229}{6144}-24\frac{(1-0)^5}{180(8)^4}=\frac{1229}{6144}-\frac{1}{30720}$$ **is true for all ξ.**

I can make a minor adjustment in my Mathematica Notebook file (change f[x] and n) to check these:

```
f[x_]:=x^4;
fPlot =
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Ident
ity];
a=0;b=1;n=4;h=(b-a)/n;points =
Partition[Flatten[Table[{{i,a},{i,f[i]},{i+b/n,f[i +
b/n]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
trapPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,trapPlot,DisplayFunction→$DisplayFunction]
```

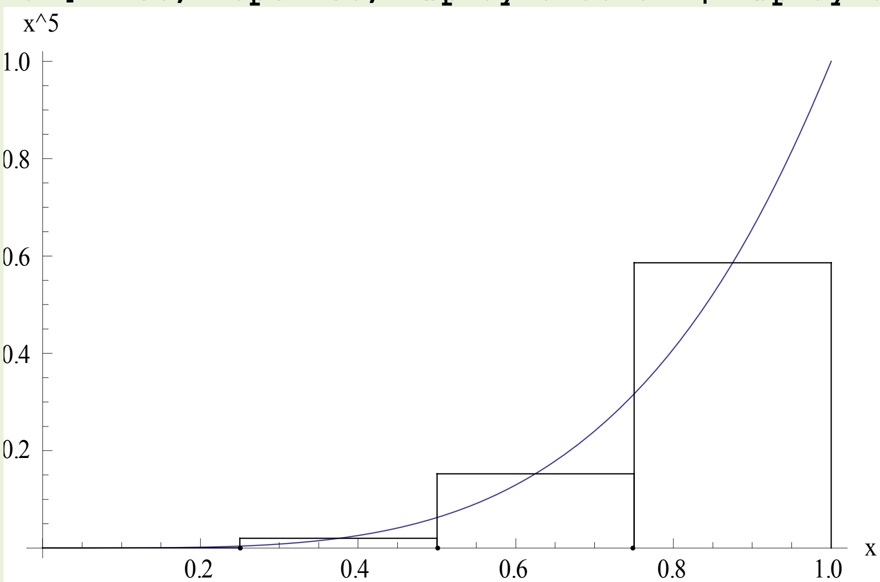



```
 trapsum = 0; Do[ trapsum = trapsum +2*f[a+i*h],{i,1,n-1}];
trapsum = trapsum + f[a]+f[b];
trapsum = h/2*trapsum; Print["The trapezoidal sum for the
integral is ", trapsum]
 The trapezoidal sum for the integral is  113/512
 Solve[Integrate[f[x],{x,0,1}]==trapsum-f''[xi]*(b-
a)^3/(12*n^2),xi]
```

$\{\{xi\to-(\sqrt{\frac{53}{10}}/4)\},\{xi\to\sqrt{\frac{53}{10}}/4\}\}$

```
 N[%]
 {{xi→-0.575543},{xi→0.575543}}
 fPlot =
```

```
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Identity];
a=0;b=1;n=4;h=(b-a)/n;points = Partition[
Flatten[Table[{{i,a},{i,f[i+b/(2*n)]},{i+b/n,f[i +
b/(2*n)]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
midptPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,midptPlot,DisplayFunction→$DisplayFunction]
```




```
midptsum = 0; Do[ midptsum = midptsum +f[a+(2*i-1)*h/2],{i,1,n}]; midptsum = h*midptsum; Print["The midpoint sum for the integral is ", midptsum]
```

The midpoint sum for the integral is 777/4096

```
Solve[Integrate[f[x],{x,0,1}]==midptsum+f''[xi]*(b-a)^3/(24*n^2),xi]
```

$\{\{xi \to -(\sqrt{\frac{211}{10}}/8)\},\{xi \to \sqrt{\frac{211}{10}}/8\}\}$

```
N[%]
```

{{xi→-0.574184},{xi→0.574184}}

```
simpsonSum = (trapsum + 2*midptsum)/3;Print["The Simpson sum for the integral is ", simpsonSum]
```

The Simpson sum for the integral is 1229/6144

```
Solve[Integrate[f[x],{x,0,1}]==simpsonSum-f''''[xi]*(b-a)^5/(180*(2*n)^4),xi]
```

{{}} (That is, it's true for all xi.)

4. $\int_0^1 x^5 dx$ **for $n = 4$.**

$$T_4 = \frac{f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)}{2}\left(\frac{1-0}{4}\right) = \frac{197}{1024}$$ **and** $f''(\xi) = 20\xi^3$**, so**

$$\int_0^1 x^5 dx = T_4 - f''(\xi)\frac{(1-0)^3}{12(4)^2}$$ **is the equation,**

$$\frac{1}{6}=\frac{197}{1024}-20\xi^3\frac{1}{192}\Leftrightarrow\frac{5\xi^3}{48}=\frac{197}{1024}-\frac{1}{6}=\frac{79}{3072}\Leftarrow\boxed{\xi=\sqrt[3]{\frac{79}{320}}=\frac{\sqrt[3]{1975}}{20}\approx 0.627324676023},$$

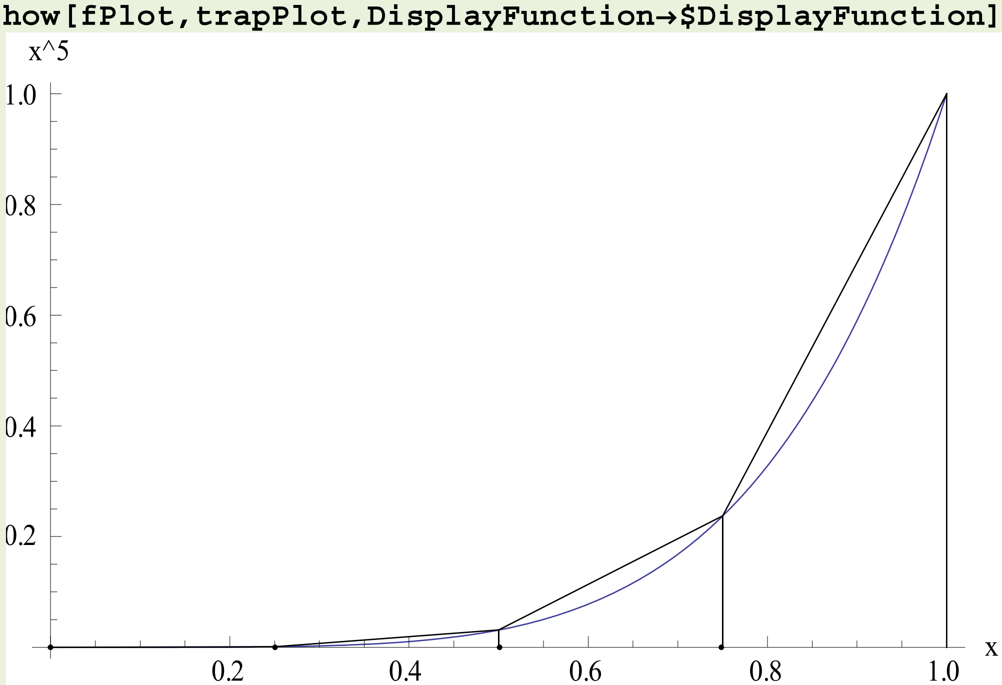
$M_4=\left(f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right)\left(\frac{1-0}{4}\right)=\frac{1261}{8192}$ **so the midpoints equation is**

$$\frac{1}{6}=\frac{1261}{8192}+\frac{5\xi^3}{96}\Leftrightarrow\frac{5\xi^3}{96}=\frac{1}{6}-\frac{1261}{8192}\Leftrightarrow\xi^3=\frac{96}{5}\left(\frac{313}{24576}\right)=\frac{313}{1280}\Leftarrow\boxed{\xi=\frac{\sqrt[3]{15650}}{40}\approx 0.625333155713}$$

$S_8=\frac{T_4+2M_4}{3}=\frac{\frac{197}{1024}+\frac{1261}{4096}}{3}=\frac{683}{4096}$**, so that**

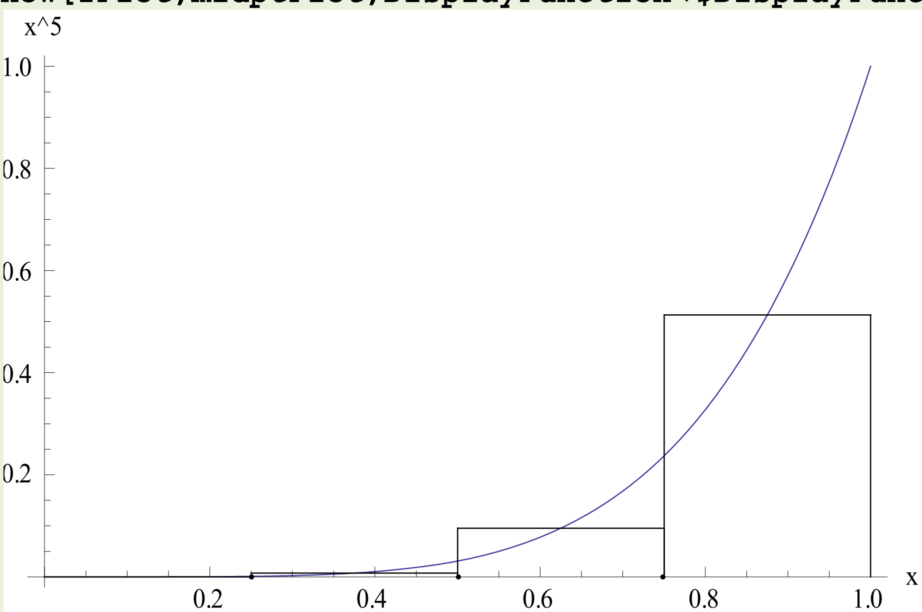
$$\frac{1}{6}=\frac{683}{4096}-120\xi\frac{(1-0)^5}{180(8)^4}=\frac{683}{4096}-\frac{\xi}{6144}\Leftrightarrow\boxed{\xi=\frac{1}{2}}.$$

```
     In Mathematica:
     f[x_]:=x^5;
fPlot =
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Ident
ity];
a=0;b=1;n=4;h=(b-a)/n;points =
Partition[Flatten[Table[{{i,a},{i,f[i]},{i+b/n,f[i +
b/n]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
trapPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,trapPlot,DisplayFunction→$DisplayFunction]
```




```
 trapsum = 0; Do[ trapsum = trapsum +2*f[a+i*h],{i,1,n-1}];
trapsum = trapsum + f[a]+f[b];
trapsum = h/2*trapsum; Print["The trapezoidal sum for the
integral is ", trapsum]
 The trapezoidal sum for the integral is  197/1024
 Solve[Integrate[f[x],{x,0,1}]==trapsum-f''[xi]*(b-
```

```
a)^3/(12*n^2),xi]
 {{xi→-(1/4) (-(79/5))^(1/3)},{xi→1/4 (79/5)^(1/3)},{xi→1/4 (-1)^(2/3)
(79/5)^(1/3)}}
 N[%]
 {{xi→-0.313662-0.543279 i},{xi→0.627325},{xi→-
0.313662+0.543279 i}}
 fPlot =
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Ident
ity];
a=0;b=1;n=4;h=(b-a)/n;points = Partition[
Flatten[Table[{{i,a},{i,f[i+b/(2*n)]},{i+b/n,f[i +
b/(2*n)]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
midptPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,midptPlot,DisplayFunction→$DisplayFunction]
```



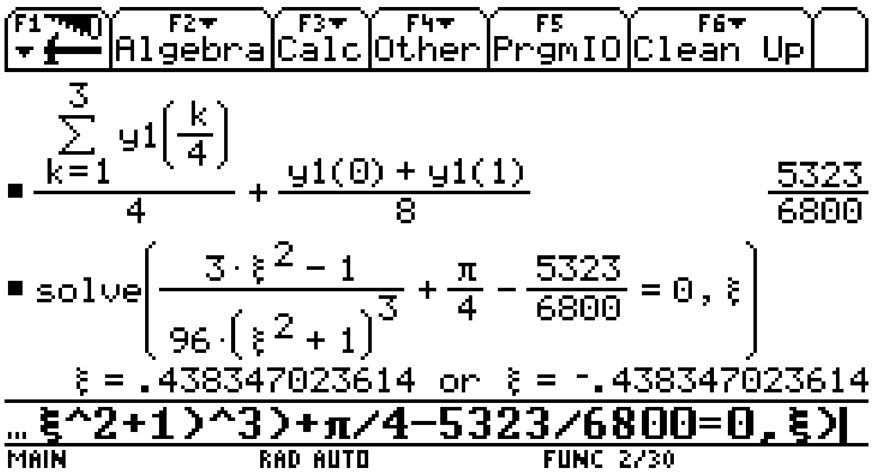

```
 midptsum = 0; Do[ midptsum = midptsum +f[a+(2*i-
1)*h/2],{i,1,n}]; midptsum = h*midptsum; Print["The midpoint
sum for the integral is ", midptsum]
 The midpoint sum for the integral is  1261/8192
 Solve[Integrate[f[x],{x,0,1}]==midptsum+f''[xi]*(b-
a)^3/(24*n^2),xi]
 {{xi→-((-(313/5))^(1/3)/(4 2^(2/3)))},{xi→(313/5)^(1/3)/(4 2^(2/3))},{xi→((-
1)^(2/3) (313/5)^(1/3))/(4 2^(2/3))}}
 N[%]
 {{xi→-0.312667-0.541554 i},{xi→0.625333},{xi→-
0.312667+0.541554 i}}
 simpsonSum = (trapsum + 2*midptsum)/3;Print["The Simpson sum
for the integral is ", simpsonSum]
 The Simpson sum for the integral is  683/4096
 Solve[Integrate[f[x],{x,0,1}]==simpsonSum-f''''[xi]*(b-
a)^5/(180*(2*n)^4),xi]
 {{xi→1/2}}
```

5. $\int_0^1 \frac{1}{x^2+1}\,dx$ for $n = 4$.

$$T_4 = \frac{f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)}{2}\left(\frac{1-0}{4}\right) = \frac{5323}{6800} \text{ and } f''(\xi) = \frac{2(3\xi^2-1)}{(\xi^2+1)^3},$$

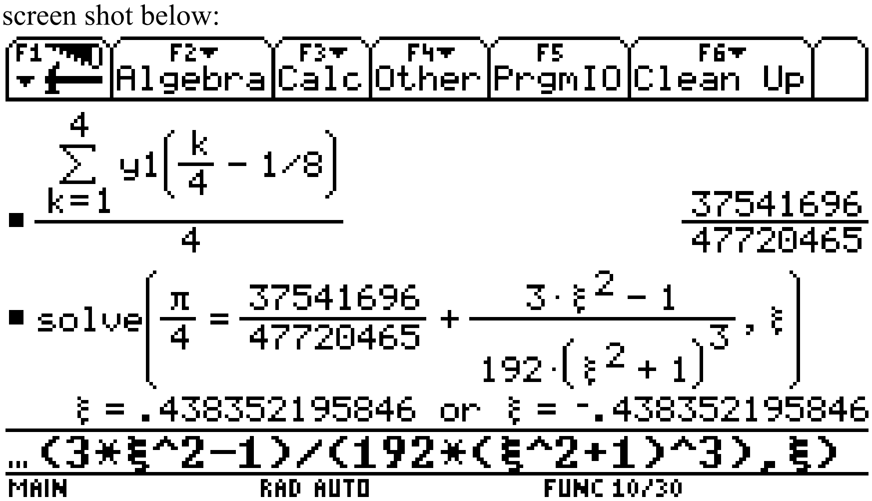
so $\int_0^1 \frac{1}{1+x^2}\,dx = T_4 - f''(\xi)\frac{(1-0)^3}{12(4)^2}$ is the equation,

$\frac{\pi}{4} = \frac{5323}{6800} - \frac{2(3\xi^2-1)}{(\xi^2+1)^3}\frac{1}{192} \Leftarrow \boxed{\xi \approx 0.438347023614}$, as computed by the TI-92 in the screen shot below:



$$M_4 = \left(f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right)\left(\frac{1-0}{4}\right) = \frac{37541696}{47720465}$$ so the midpoints equation is

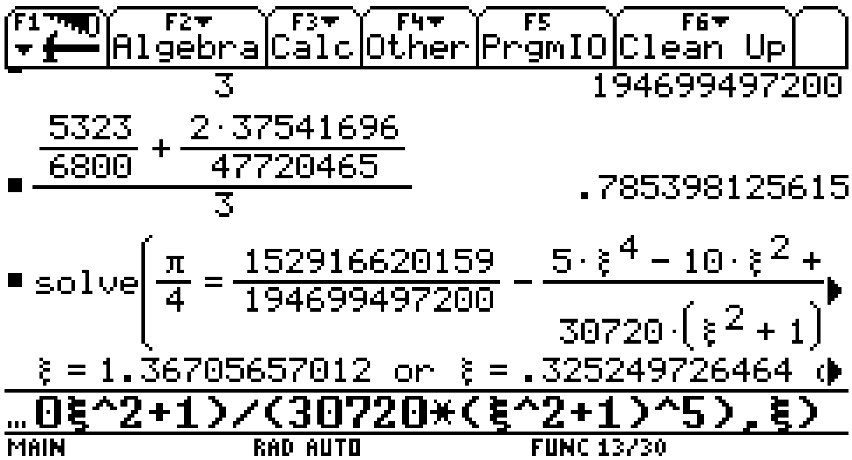
$\frac{\pi}{4} = \frac{37541696}{47720465} + \frac{3\xi^2-1}{192(\xi^2+1)^3} \Leftarrow \boxed{\xi \approx 0.438352195846}$, as computed by the TI-92 in the screen shot below:



$$S_8 = \frac{T_4 + 2M_4}{3} = \frac{\frac{5323}{6800} + \frac{75083392}{47720465}}{3} = \frac{152916620159}{194699497200} \approx 0.785398125615 \text{, so that}$$

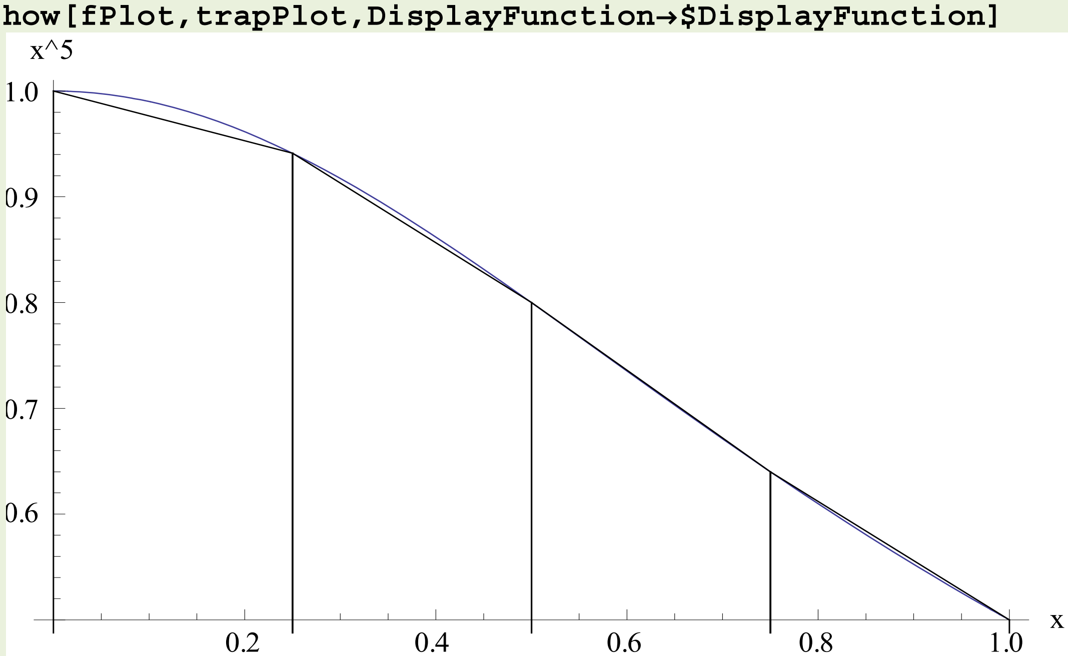
$$\frac{\pi}{4} = \frac{152916620159}{194699497200} - \frac{5\xi^4 - 10\xi^2 + 1}{30720\left(\xi^2+1\right)^5} \Leftarrow \boxed{\xi = 0.325249726464}.$$

as computed by the TI-92 in the screen shot following:

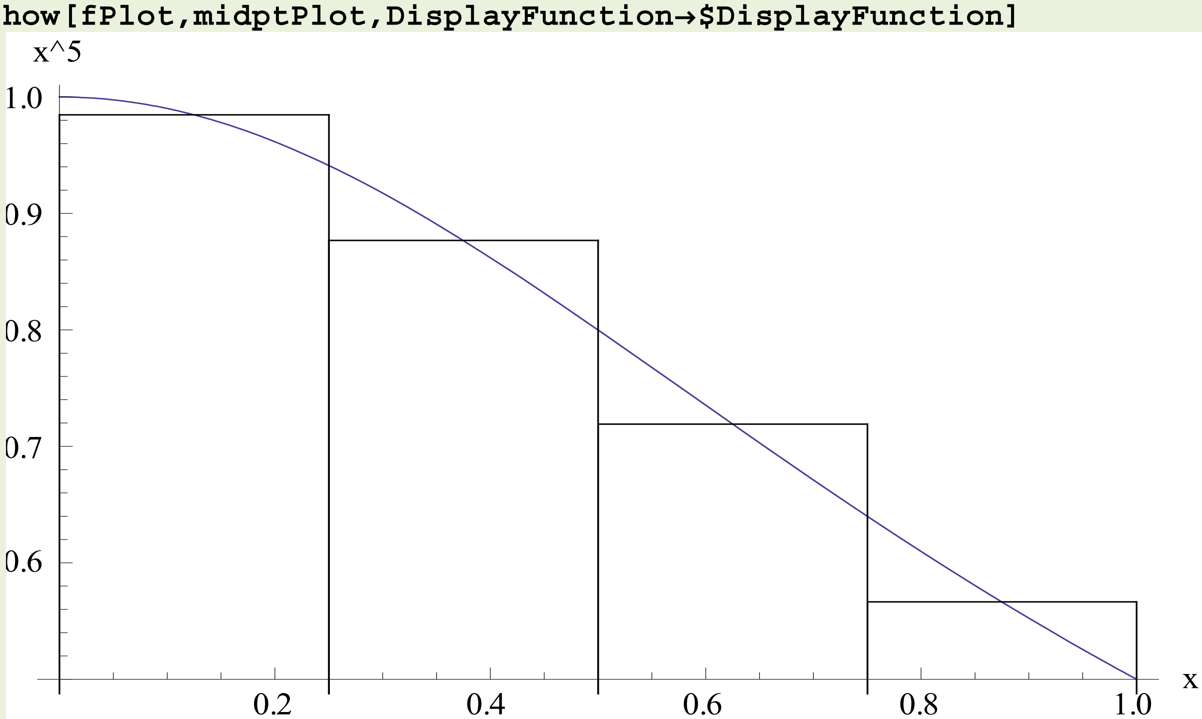


Or, in Mathematica:

```
f[x ]:=1/(1+x^2);
fPlot =
Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Identity];
a=0;b=1;n=4;h=(b-a)/n;points =
Partition[Flatten[Table[{{i,a},{i,f[i]},{i+b/n,f[i +
b/n]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
trapPlot=Show[Graphics[Line /@ Table[Take[points, {i,i+1}],{i,1,
Length[points]-1}]], DisplayFunction→Identity];
Show[fPlot,trapPlot,DisplayFunction→$DisplayFunction]
```



```
trapsum = 0; Do[ trapsum = trapsum +2*f[a+i*h],{i,1,n-1}];
trapsum = trapsum + f[a]+f[b];
trapsum = h/2*trapsum; Print["The trapezoidal sum for the integral is ", trapsum]
 The trapezoidal sum for the integral is  5323/6800
 Solve[Integrate[f[x],{x,0,1}]==trapsum-f''[xi]*(b-a)^3/(12*n^2),xi]
 {{xi→-\[Sqrt](-1+1/(18 (-5323+1700 π) (2/(46819645239600-29905465680000 π+4775436000000 π^2+\[Sqrt](2184786663206464393611660000-2793339580869503401806000000 π+133927388447203309860000 [..a lot of numbers omitted here].
 N[%]
 {{xi→-0.438347},{xi→0.438347},{xi→-1.08521-1.66546 ⅈ},{xi→1.08521+1.66546 ⅈ},{xi→-1.08521+1.66546 ⅈ},{xi→1.08521 -1.66546 ⅈ}}
 fPlot = Plot[f[x],{x,0,1},AxesLabel→{"x","x^5"},DisplayFunction→Identity];
a=0;b=1;n=4;h=(b-a)/n;points = Partition[ Flatten[Table[{{i,a},{i,f[i+b/(2*n)]},{i+b/n,f[i + b/(2*n)]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
midptPlot=Show[Graphics[Line /@ Table[Take[points, {i,i+1}],{i,1, Length[points]-1}]], DisplayFunction→Identity];
Show[fPlot,midptPlot,DisplayFunction→$DisplayFunction]
```




```
 midptsum = 0; Do[ midptsum = midptsum +f[a+(2*i-1)*h/2],{i,1,n}];
midptsum = h*midptsum; Print["The midpoint sum for the integral is ", midptsum]
 The midpoint sum for the integral is  37541696/47720465
 Solve[Integrate[f[x],{x,0,1}]==midptsum+f''[xi]*(b-a)^3/(24*n^2),xi]
 {{xi→-\[Sqrt](-1+1/(144 (-150166784+47720465 π) (2/(-267767989028713592612223713280+170184279215370437596584345600 π-27040843266135713572192128000 π^2+\[Sqrt][..omitted stuff..]
 N[%]
 {{xi→-0.438365},{xi→0.438365},{xi→-1.08523+1.66547 ⅈ},{xi→1.08523
```

```
-1.66547 ⅈ},{xi→-1.08523-1.66547 ⅈ},{xi→1.08523 +1.66547 ⅈ}}
 simpsonSum = (trapsum + 2*midptsum)/3;Print["The Simpson sum for
the integral is ", simpsonSum]
 The Simpson sum for the integral is  152916620159/194699497200
 Solve[Integrate[f[x],{x,0,1}]==simpsonSum-f''''[xi]*(b-
a)^5/(180*(2*n)^4),xi]
 {{xi→-\[Sqrt]Root[-19572516132447+6230383910400 π+(-
97874749380810+31151919552000 π)
N[%]
{{xi→0. -4.29856 ⅈ},{xi→0. +4.29856 ⅈ},{xi→-0.32525},{xi→0.32525},{xi→-
1.36706},{xi→1.36706},{xi→-3.24944+2.19258 ⅈ},{xi→3.24944 -2.19258 ⅈ},{xi→-3.24944-2.19258
ⅈ},{xi→3.24944 +2.19258 ⅈ}}
```

6. $\int_0^{\pi/2} \sin^4(x)\,dx$ **for $n = 4$.**

$T_4 = \dfrac{f(0)+2f(\pi/8)+2f(\pi/4)+2f(3\pi/8)+f(\pi/2)}{2}\left(\dfrac{\pi/2-0}{4}\right) = \dfrac{3\pi}{16}$ **and**

$f''(\xi) = 4\sin^2\xi\left(3\cos^2\xi - \sin^2\xi\right)$, **so** $\int_0^{\pi/2}\sin^4 x\,dx = T_4 - f''(\xi)\dfrac{(\pi/2-0)^3}{12(4)^2}$ **is the**

equation, $\dfrac{3\pi}{16} = \dfrac{3\pi}{16} - \left(12\sin^2\xi\cos^2\xi - 4\sin^4\xi\right)\dfrac{\pi^3}{8} \Leftarrow \dfrac{3}{4} = \sin^2\xi \Leftarrow \boxed{\xi = \dfrac{\pi}{3}}$

$M_4 = \left(f\left(\dfrac{\pi}{16}\right)+f\left(\dfrac{3\pi}{16}\right)+f\left(\dfrac{5\pi}{16}\right)+f\left(\dfrac{7\pi}{16}\right)\right)\left(\dfrac{\pi/2-0}{4}\right) = \dfrac{3\pi}{16}$ **so the midpoints equation is**

$\dfrac{3\pi}{16} = \dfrac{3\pi}{16} + \left(12\sin^2\xi\cos^2\xi - 4\sin^4\xi\right)\dfrac{\pi^3}{8\cdot 384} \Leftarrow \dfrac{3}{4} = \sin^2\xi \Leftarrow \boxed{\xi = \dfrac{\pi}{3}}$

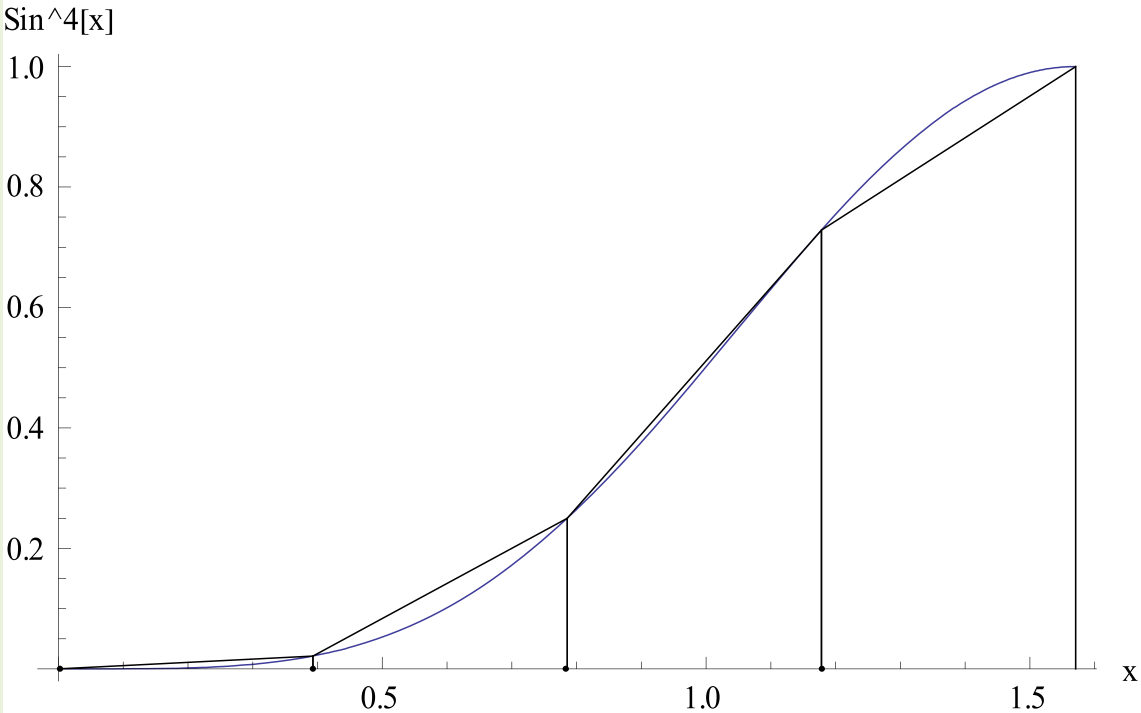
$S_8 = \dfrac{T_4 + 2M_4}{3} = \dfrac{3\pi}{16}$**, so that**

$\dfrac{3\pi}{16} = \dfrac{3\pi}{16} - \left[\left(24 - 216\sin^2\xi\right)\cos^2\xi + 40\sin^4\xi\right]\dfrac{\pi^5}{32\cdot 180\cdot 8^4} \Leftrightarrow 8\cos^2(2\xi) - \cos(2\xi) - 4 = 0$.

$\Leftrightarrow \cos(2\xi) = \dfrac{1\pm\sqrt{129}}{16} \Leftarrow \boxed{\xi = 0.344121245418}$

In Mathematica:

```
f[x_]:=(Sin[x])^4;
fPlot =
Plot[f[x],{x,0,Pi/2},AxesLabel→{"x","Sin^4[x]"},DisplayFuncti
on→Identity];
a=0;b=Pi/2;n=4;h=(b-a)/n;points =
Partition[Flatten[Table[{{i,a},{i,f[i]},{i+b/n,f[i +
b/n]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
trapPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,trapPlot,DisplayFunction→$DisplayFunction]
```




```
trapsum = 0; Do[ trapsum = trapsum +2*f[a+i*h],{i,1,n-1}];
trapsum = trapsum + f[a]+f[b];
trapsum = h/2*trapsum; Print["The trapezoidal sum for the
integral is ", trapsum]
```

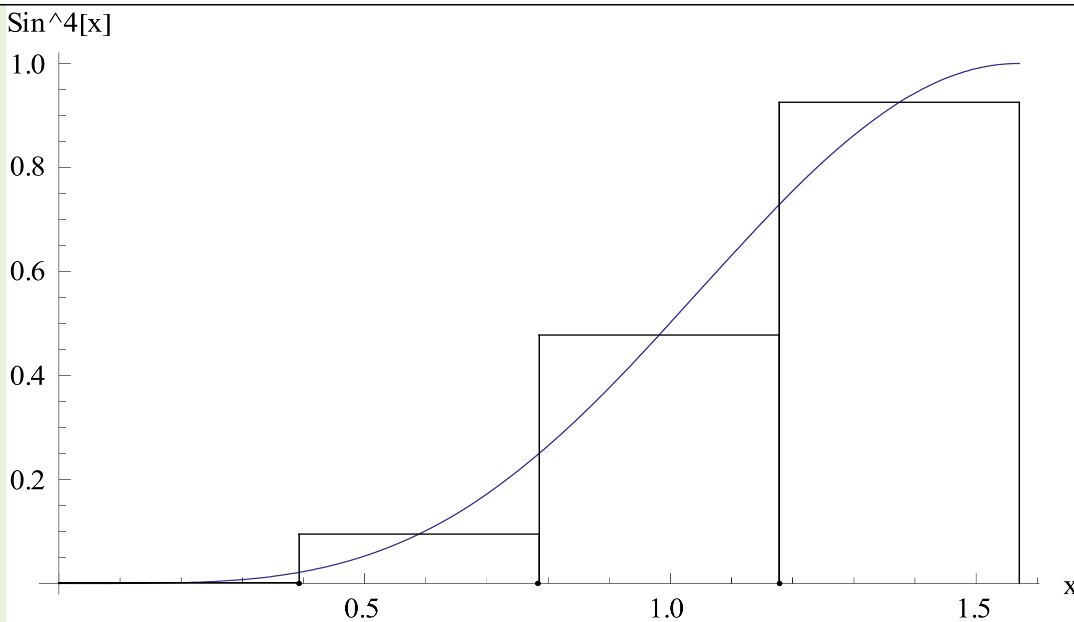
```
The trapezoidal sum for the integral is  1/16 π (3/2+2
Cos[π/8]^4+2 Sin[π/8]^4)
```

```
Solve[Integrate[f[x],{x,0,1}]==trapsum-f''[xi]*(b-
a)^3/(12*n^2),xi]
```

Solve::ifun: Inverse functions are being used by \[NoBreak]Solve\[NoBreak], so some solutions may not be found; use Reduce for complete solution information. ≫

```
{{xi→-ArcCos[-(1/2) \[Sqrt](1/2 (5-(1/π3/2)(\[Sqrt](3 (768-
192 π+3 π3-256 π Cos[π/8]^4-512 Sin[2]+64 Sin[4]-256 π
Sin[π/8]^4))))))]}, [..weird output..]
```

```
fPlot =
Plot[f[x],{x,0,Pi/2},AxesLabel→{"x","Sin^4[x]"},DisplayFuncti
on→Identity];
a=0;b=Pi/2;n=4;h=(b-a)/n;points = Partition[
Flatten[Table[{{i,a},{i,f[i+b/(2*n)]},{i+b/n,f[i +
b/(2*n)]},{i+b/n,0}},{i,0,((n-1)/n)*b,b/n}]],2];
midptPlot=Show[Graphics[Line /@ Table[Take[points,
{i,i+1}],{i,1, Length[points]-1}]],
DisplayFunction→Identity];
Show[fPlot,midptPlot,DisplayFunction→$DisplayFunction]
```




```
midptsum = 0; Do[ midptsum = midptsum +f[a+(2*i-1)*h/2],{i,1,n}]; midptsum = h*midptsum; Print["The midpoint sum for the integral is ", midptsum]
```

The midpoint sum for the integral is 1/8 π (Cos[π/16]^4+Cos[(3 π)/16]^4+Sin[π/16]^4+Sin[(3 π)/16]^4)

```
Solve[Integrate[f[x],{x,0,1}]==midptsum+f''[xi]*(b-a)^3/(24*n^2),xi]
```

Solve::ifun: Inverse functions are being used by \[NoBreak]Solve\[NoBreak], so some solutions may not be found; use Reduce for complete solution information. ≫

{{xi→-ArcCos[-(1/2) \[Sqrt](1/2 (5-(1/π3/2)(\[Sqrt](3 (-1536+3 π3+512 π Cos[π/16]^4+512 π Cos[(3 π)/16]^4+1024 Sin[2]-128 Sin[4]+512 π Sin[π/16]^4+512 π Sin[(3 π)/16]^4))))))]}, [..weird output..]

```
N[%]
```

{{xi→-1.5708-0.919683 ⅈ},{xi→1.5708 +0.919683 ⅈ},{xi→-1.5708+0.919683 ⅈ},{xi→1.5708 -0.919683 ⅈ},{xi→-3.14159+0.994947 ⅈ},{xi→3.14159 -0.994947 ⅈ},{xi→0. -0.994947 ⅈ},{xi→0. +0.994947 ⅈ}}

```
simpsonSum = (trapsum + 2*midptsum)/3;Print["The Simpson sum for the integral is ", simpsonSum]
```

The Simpson sum for the integral is 1/3 (1/16 π (3/2+2 Cos[π/8]^4+2 Sin[π/8]^4)+1/4 π (Cos[π/16]^4+Cos[(3 π)/16]^4+Sin[π/16]^4+Sin[(3 π)/16]^4))

```
Solve[Integrate[f[x],{x,0,1}]==simpsonSum-f''''[xi]*(b-a)^5/(180*(2*n)^4),xi]
```

Solve::ifun: Inverse functions are being used by \[NoBreak]Solve\[NoBreak], so some solutions may not be found; use Reduce for complete solution information. ≫

{{xi→-ArcCos[-(1/4) \[Sqrt](1/2 (17-(1/π5/2)(\[Sqrt](3 (-11796480+983040 π+43 π5+2621440 π Cos[π/16]^4+ [..weird output..]

```
N[%]
```

{{xi→-1.5708-1.92732 ⅈ},{xi→1.5708 +1.92732 ⅈ},{xi→-1.5708+1.92732 ⅈ},{xi→1.5708 -1.92732 ⅈ},{xi→-3.14159+1.92996 ⅈ},{xi→3.14159 -1.92996 ⅈ},{xi→0. -1.92996 ⅈ},{xi→0. +1.92996 ⅈ}}

	Tn	Mn	S2n
1. x^4 n=2 0<x<1	T2 = 9/32 $\frac{1}{5}=\frac{9}{32}-\frac{y''}{48}$ $\frac{\xi^2}{4}=\frac{13}{160}\Leftarrow$ $\boxed{\xi=\sqrt{\frac{13}{40}}=\frac{\sqrt{130}}{20}\approx 0.570088}$	$\frac{1}{5}=\frac{41}{256}+12\xi^2\frac{(1-0)^3}{24(2)^2}$ $\frac{\xi^2}{8}=\frac{1}{5}-\frac{41}{256}$ $\boxed{\xi=\frac{\sqrt{510}}{40}\approx 0.564579}$	$\frac{1}{5}=\frac{77}{384}-24\frac{(1-0)^5}{180(4)^4}$ $\frac{77}{384}-\frac{2}{3840}=\frac{768}{3840}=\frac{1}{5}$ True for all ξ
2. x^5 n=2 0<x<1	$\frac{1}{6}=\frac{17}{64}-20\xi^3\frac{1}{48}$ $\frac{5\xi^3}{12}=\frac{19}{192}$ $\boxed{\xi=\sqrt[3]{\frac{19}{80}}\approx 0.619281}$	$\frac{1}{6}=\frac{61}{512}+\frac{5\xi^3}{24}$ $\xi^3=\frac{73}{320}$ $\boxed{\xi=\frac{\sqrt[3]{1825}}{20}\approx 0.611023}$	$\frac{1}{6}=\frac{43}{256}-120\xi\frac{(1-0)^5}{180(4)^4}$ $\frac{43}{256}-\frac{\xi}{384}\Leftrightarrow\boxed{\xi=\frac{1}{2}}$
3. x^4 n=4 0<x<1	$\frac{1}{5}=\frac{113}{512}-\xi^2\frac{1}{16}$ $\frac{\xi^2}{16}=\frac{53}{2560}$ $\boxed{\xi=\sqrt{\frac{530}{1600}}\approx 0.575543}$	$\frac{1}{5}=\frac{777}{4096}+\frac{\xi^2}{32}$ $\xi^2=\frac{211}{640}$ $\boxed{\xi=\frac{\sqrt{2110}}{80}\approx 0.574184}$	$\frac{1}{5}=\frac{1229}{6144}-24\frac{(1-0)^5}{180(8)^4}$ $=\frac{1229}{6144}-\frac{1}{30720}$ True for all ξ
4. x^5 n=4 0<x<1	$\frac{1}{6}=\frac{197}{1024}-20\xi^3\frac{1}{192}$ $\frac{5\xi^3}{48}=\frac{197}{1024}-\frac{1}{6}=\frac{79}{3072}$ $\boxed{\xi=\sqrt[3]{\frac{79}{320}}\approx 0.627325}$	$\frac{1}{6}=\frac{1261}{8192}+\frac{5\xi^3}{96}$ $\xi^3=\frac{313}{1280}$ $\boxed{\xi=\frac{\sqrt[3]{15650}}{40}\approx 0.625333}$	$\frac{1}{6}=\frac{683}{4096}-120\xi\frac{(1-0)^5}{180(8)^4}$ $=\frac{683}{4096}-\frac{\xi}{6144}$ $\Leftrightarrow\boxed{\xi=\frac{1}{2}}$
5. $1/(x^2+1)$ 0<x<1	$\frac{\pi}{4}=\frac{5323}{6800}-\frac{2(3\xi^2-1)}{(\xi^2+1)^3}\frac{1}{192}$ $\Leftarrow\boxed{\xi\approx 0.438347}$	$\frac{\pi}{4}=\frac{37541696}{47720465}+\frac{3\xi^2-1}{192(\xi^2+1)^3}$ $\Leftarrow\boxed{\xi\approx 0.438352}$	$\frac{\pi}{4}=\frac{152916620159}{194699497200}-\frac{5\xi^4-10\xi^2+1}{30720(\xi^2+1)^5}$ $\Leftarrow\boxed{\xi=0.325249726464}$
6. $\sin^4(x)$ 0<x<π/2	$0=(12\sin^2\xi\cos^2\xi-4\sin^4\xi)\frac{\pi^3}{8}$ $\Leftarrow\frac{3}{4}=\sin^2\xi\Leftarrow\boxed{\xi=\frac{\pi}{3}}$	$0=(12\sin^2\xi\cos^2\xi-4\sin^4\xi)\frac{\pi^3}{8\cdot 384}$ $\frac{3}{4}=\sin^2\xi\Leftarrow\boxed{\xi=\frac{\pi}{3}}$	$(216\sin^2\xi-24)\cos^2\xi=40\sin^4\xi$ $8\cos^2(2\xi)-\cos(2\xi)-4=0$ $\cos(2\xi)=\frac{1\pm\sqrt{129}}{16}$ $\boxed{\xi=0.344121}$