Math 1B – Calculus II – Test #2 – Spring '10 Name________________________________

Show your work for credit. Write all responses on separate paper. Do not use a calculator.

1. The voltage V at time t of house current is given by $V(t) = C\sin(120\pi t)$ where t is in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of oscillation (one cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C.

2. The solid torus is the figure obtained by rotating the disk $(x-b)^2 + y^2 \le a^2$ around the y-axis. Find its volume by the method of shells. (Hint: Substitute for $x - b$. Note that the answer is the area of the disk multiplied by the distance travelled by the center as it revolves.)

3. For any integer $n \ge 0$, consider $\int \tan^{n+2} x dx$

   a. Use the substitution $\tan^2 x = \sec^2 x - 1$ to show that $\int \tan^{n+2} x dx = \dfrac{1}{n+1}\tan^{n+1} x - \int \tan^n x dx$

   b. Simplify $\int \tan^4 x dx$

4. Consider $\int \sec x\, dx$

   a. Show that $\sec x = \dfrac{\cos x}{1-\sin^2 x}$

   b. Substitute $\sec x = \dfrac{\cos x}{1-\sin^2 x}$ into $\int \sec x\, dx$, then substitute for $\sin x$ and simplify the integral using partial fractions. Write your final answer in terms of x.

   c. Convert the formula into the more familiar one by multiplying the fraction in the answer by $1 = \dfrac{1+\sin x}{1+\sin x}$ (recall that $(1/2)\ln u = \ln\sqrt{u}$)

5. Consider the volume generated by rotating the area below $y = \cos x$ for $0 \le x \le \dfrac{\pi}{2}$ about the y-axis.

   a. Compute the volume by the method of shells.

   b. Show that the height h to which the volume must be filled to be half full (by volume) satisfies the equation $4h\sin(h) + 4\cos(h) - 2 - \pi = 0$

6. A container is formed by revolving the curve $y = x^2$ for $0 \le x \le 4$ about the y-axis (where x and y are both measured in meters.) How much work is required to pump a fluid with weight density 1000N/m^3 out the top?

7. Simplify $I = \int_0^x e^u \sin u\, du$. Hint: use integration by parts twice so that the integrand recurs, then solve for I.

8. Simplify $\displaystyle\int_0^1 \frac{x^2+3x+3}{(x^2+2x+2)(x+1)}dx$

Math 1B – Calculus II – Test #2 Solutions – Spring '10

1. The voltage V at time t of house current is given by $V(t) = C\sin(120\pi t)$ where t is in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of oscillation (one cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C.

   SOLN: The period of oscillation can be found by computing t so that $0 \le 120\pi t \le 2\pi \Leftrightarrow 0 \le t \le \frac{1}{60}$

   Then RMS =

   $$\sqrt{\frac{C^2}{1/60-0}\int_0^{1/60}\sin^2 120\pi t\,dt} = C\sqrt{60\int_0^{1/60}\frac{1-\cos 240\pi t}{2}dt} = C\sqrt{30t - \frac{\sin 240\pi t}{8}\Big|_0^{1/60}} = C\sqrt{\frac{1}{2}-0} = \frac{C\sqrt{2}}{2}$$

2. The solid torus is the figure obtained by rotating the disk $(x-b)^2 + y^2 \le a^2$ around the y-axis. Find its volume by the method of shells. (Hint: Substitute for $x - b$. Note that the answer is the area of the disk multiplied by the distance travelled by the center as it revolves.)

   SOLN: We seek r and h in $V = 2\pi\int_{b-a}^{b+a} rh\,dx$. The radius $r = x$ and the height $h = 2y = 2\sqrt{a^2-(x-b)^2}$ so that $V = 4\pi\int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}\,dx$. Substituting $u = x - b$ we get

   $$V = 4\pi\int_{-a}^{a}(u+b)\sqrt{a^2-u^2}\,du = 4\pi\int_{-a}^{a}u\sqrt{a^2-u^2}\,du + 4\pi b\int_{-a}^{a}\sqrt{a^2-u^2}\,du$$

   The first integral has an odd integrand over an interval centered on the origin, so it's zero.
   The second integral is half the area of a circle of radius a, so $V = (2\pi b)(\pi a^2)$

3. For any integer $n \ge 0$, consider $\int \tan^{n+2} x\,dx$

   a. Use the substitution $\tan^2 x = \sec^2 x - 1$ to show that $\int \tan^{n+2} x\,dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\,dx$

   SOLN:
   $$\int \tan^{n+2}(x)\,dx = \int \tan^2(x)\tan^n(x)\,dx = \int\left(\sec^2(x)-1\right)\tan^n(x)\,dx$$
   $$= \int \sec^2(x)\tan^n(x)\,dx - \int \tan^n(x)\,dx$$

   Substitute $u = \tan x$ so that $du = \sec^2(x)\,dx$ and the integral is then

   $$\int u^n\,du - \int \tan^n x\,dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\,dx$$

   b. Simplify $\int \tan^4 x\,dx$

   SOLN: $\int \tan^4(x)\,dx = \frac{\tan^3 x}{3} - \int \tan^2(x)\,dx = \frac{\tan^3 x}{3} - \tan x + \int dx = \frac{\tan^3 x}{3} - \tan x + x + c$

4. Consider $\int \sec x\,dx$

a. Show that $\sec x = \dfrac{\cos x}{1-\sin^2 x}$

SOLN: $\sec x = \dfrac{1}{\cos x} = \dfrac{\cos x}{\cos^2 x} = \dfrac{\cos x}{1-\sin^2 x}$

b. Substitute $\sec x = \dfrac{\cos x}{1-\sin^2 x}$ into $\int \sec x\,dx$, then substitute for $\sin x$ and simplify the integral using partial fractions. Write your final answer in terms of *x*.

SOLN:
$$\int \sec x\,dx = \int \frac{\cos x}{1-\sin^2 x}dx = \int \frac{du}{1-u^2} = \int \frac{du}{(1+u)(1-u)} = \frac{1}{2}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{1-u}$$
$$= \frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| + c = \frac{1}{2}\ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x| + c$$

c. Convert the formula into the more familiar one by multiplying the fraction in the answer by $1 = \dfrac{1+\sin x}{1+\sin x}$

(recall that $(1/2)\ln u = \ln\sqrt{u}$)

SOLN:
$$\int \sec x\,dx = \ln\sqrt{|1+\sin x|} - \ln\sqrt{|1-\sin x|} + c = \ln\sqrt{\left|\frac{1+\sin x}{1-\sin x}\right|} + c = \ln\sqrt{\left|\frac{1+\sin x}{1-\sin x}\right|\left|\frac{1+\sin x}{1+\sin x}\right|} + c$$
$$= \ln\sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}} + c = \ln\sqrt{\left(\frac{1+\sin x}{\cos x}\right)^2} + c = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c = \ln|\sec x + \tan x| + c$$

5. Consider the volume generated by rotating the area below $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$ about the *y*-axis.

a. Compute the volume by the method of shells.

SOLN:
$$2\pi\int_0^{\pi/2} rh\,dx = 2\pi\int_0^{\pi/2} x\cos x\,dx = 2\pi\left(x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right) = 2\pi\left(\frac{\pi}{2} + \cos x\Big|_0^{\pi/2}\right)$$
$$= 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi = \pi(\pi-2)$$

b. Show that the height *h* to which the volume must be filled to be half full (by volume) satisfies the equation $4h\sin(h) + 4\cos(h) - 2 - \pi = 0$

SOLN:
$$2\pi\int_0^{h} x\cos x\,dx = \frac{\pi(\pi-2)}{2} \Leftrightarrow x\sin x + \cos x\Big|_0^h = \frac{\pi-2}{4} \Leftrightarrow h\sin(h) + \cos(h) - 1 = \frac{\pi-2}{4}$$
$$\Leftrightarrow 4h\sin(h) + 4\cos(h) - 2 - \pi = 0$$

6. A container is formed by revolving the curve $y = x^2$ for $0 \le x \le 4$ about the *y*-axis (where *x* and *y* are both measured in meters.) How much work is required to pump a fluid with weight density 1000N/m^3 out the top?

SOLN:
$$W = \int dW = 1000\int dF = 1000\int (16-y)\,dV = 1000\pi\int_0^4 (16-y)\,y\,dy = 1000\pi\left(8y^2 - \frac{y^3}{3}\right)\Bigg|_0^4$$
$$= 1000\pi\left(128 - \frac{64}{3}\right) = \frac{320000\pi}{3} \text{ Joules}$$

7. Simplify $I=\int_0^x e^u \sin u\,du$. Hint: use integration by parts twice so that the integrand recurs, then solve for *I*.

SOLN:

$$I=\int_0^x e^u \sin u\,du = e^x \sin x - 0 - \int_0^x e^u \cos u\,du = e^x \sin x - e^x \cos x + 1 - \int_0^x e^u \sin u\,du$$

$$2I = e^x \sin x - e^x \cos x + 1 \Leftrightarrow I = \frac{1}{2}\left(e^x \sin x - e^x \cos x + 1\right)$$

8. Simplify $\int_0^1 \frac{x^2+3x+3}{\left(x^2+2x+2\right)\left(x+1\right)}dx$

SOLN:

$$\int_0^1 \frac{x^2+3x+3}{\left(x^2+2x+2\right)\left(x+1\right)}dx = \int_0^1 \frac{1}{x^2+2x+2}+\frac{1}{x+1}dx = \int_0^1 \frac{1}{\left(x+1\right)^2+1}dx + \ln\left|x+1\right|\Big|_0^1$$

$$= \arctan\left(x+1\right)\Big|_0^1 + \ln 2 = \boxed{\arctan\left(2\right)-\frac{\pi}{4}+\ln 2}$$