Math 1B – Test 2 Make-up exam for Aubrey Howard

1. Find the volume generated by rotating the region bounded by $y = \sin(\pi x)$ between $0 \le x \le \pi$ about the line $x = 2$.

2. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we have $v_{\text{avg}} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s, we have $v_{\text{avg}} = \frac{2}{3}v_T$.

3. Compute the partial fraction decomposition of $f(x) = \frac{1}{(x^2-1)^2}$. Show all your work.

4. Use polynomial division to compute the following indefinite integral (for x > 1): $\int \frac{x^3-1}{(x-1)^2} dx$

5. This problem computes a recursive relation for the antiderivative of $\sec^n(\theta)$
   a. Compute the derivative of $\sec^{n-2}(\theta)\tan(\theta)$. Eliminate $\tan(\theta)$ from your answer using a trigonometry identity in order to express the derivative in terms of $\sec(\theta)$ alone. Show all your work
   b. Write your answer from (a) in the form of an integral identity, and then solve for $\int \sec^n(\theta)\, d\theta$ to find a recursive formula in the form
   $$\int \sec^n(\theta)\, d\theta = F(\theta) + A\int \sec^{n-2}(\theta)\, d\theta$$
   c. Use your recursive formula to compute the antiderivative of $\int \sec^4(\theta)\, d\theta$

6. Consider $\int \frac{1}{x^2\sqrt{x^2-1}} dx$
   a. Use a trigonometric substitution of the form $x = f(\theta)$ to express the following indefinite integral in terms of the variable θ (assume $x > 1$).
   b. Back substitute to express the integral in terms of x.

7. In each of the following, use completing the square and a trigonometric substitution to evaluate the indefinite integral.
   a. $\int \frac{1}{\sqrt{5x-x^2}} dx$ Assume that $5x - x^2 \ge 0$
   b. $\int \frac{1}{x^2-6x+18} dx$

8. Suppose the volume created by rotating the curve $y = e^x$ (where $0 \le x \le \ln 2$) about the y-axis is filled with a fluid whose force density is 1000 N/m^3. Find the work required to pump that fluid out the top of the container.

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1. Find the volume generated by rotating the region bounded by $y = \sin(\pi x)$ between $0 \le x \le \pi$ about the line $x = 2$.
   SOLN: A bit awkward, since the part to the right of $x = 2$ doesn't overlap the part to the left. All together,

$$2\pi\int rh\,dx = 2\pi\int_0^2 (2-x)2\left|\sin(\pi x)\right|dx$$

$$= 4\pi\int_0^1 (2-x)\sin(\pi x)\,dx + 4\pi\int_1^2 (x-2)\sin(\pi x)\,dx$$

you have

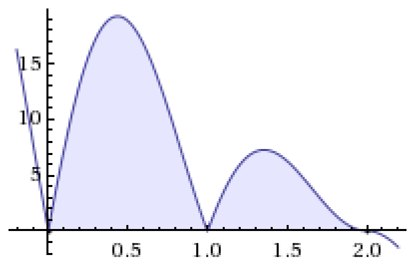
$$= -8\cos(\pi x)\Big|_0^1 + 4x\cos(\pi x)\Big|_0^1 - 4\int_0^1 \cos(\pi x)\,dx$$

$$+8\cos(\pi x)\Big|_1^2 - 4x\cos(\pi x)\Big|_1^2 + 4\int_1^2 \cos(\pi x)\,dx$$

$$= 16 - 4 + 16 - 12 = 16$$

This is from the Wolfram Alpha site. You can see how the contribution to volume increases with radius and height.

$$\int_0^2 4\pi(2-x)\left|\sin(\pi x)\right|dx = 16$$



2. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$.
   Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we have $v_{\text{avg}} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s, we have $v_{\text{avg}} = \frac{2}{3}v_T$.

   SOLN: The time averaged velocity is $\frac{1}{T-0}\int_0^T \frac{ds}{dt}dt = \frac{1}{T}\int_0^T gt\,dt = \frac{1}{T}\frac{gt^2}{2}\Big|_0^T = \frac{gT}{2} = \frac{v_T}{2}$

   The velocity is $\frac{ds}{dt} = gt = g\sqrt{\frac{2s}{g}} = \sqrt{2sg}$. So the distance averaged velocity is

$$\frac{1}{gT^2/2-0}\int_0^{gT^2/2}\sqrt{2gs}\,ds = \frac{2\sqrt{2g}}{gT^2}\int_0^{gT^2/2}\sqrt{s}\,ds = \frac{4\sqrt{2g}}{3gT^2}s^{3/2}\Big|_0^{gT^2/2} = \frac{4\sqrt{2g}}{3gT^2}\frac{gT^2}{2}\sqrt{\frac{gT^2}{2}} = \frac{2gT}{3} = \frac{2v_T}{3}$$

3. Compute the partial fraction decomposition of $f(x) = \frac{1}{(x^2-1)^2}$. Show all your work.

   SOLN: $$\frac{1}{(x^2-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$1 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2$$

   Plugging in $x = 1$, we get $B = ¼$ and plugging in $x = -1$ we get $D = ¼$. Plugging in $x = 0$ then yields $C - A = ½$ and then $x = 2$ gives $9A + 9/4 + 3C + ¼ = 1$. Substituting $C = A + ½$ gives us $12A + 4 = 1$ so $A = -¼$ and $C = ¼$ and $\frac{1}{(x^2-1)^2} = \frac{-1/4}{x-1} + \frac{1/4}{(x-1)^2} + \frac{1/4}{x+1} + \frac{1/4}{(x+1)^2}$

4. Use polynomial division to compute the following indefinite integral (for x > 1): $\int \frac{x^3-1}{(x-1)^2}dx$

SOLN: $\frac{x^3-1}{(x-1)^2} = \frac{(x-1)(x^2+x+1)}{(x-1)^2} = \frac{x^2+x+1}{x-1} = x+2+\frac{3}{x-1}$ so

$$\int \frac{x^3-1}{(x-1)^2}dx = \int x+2+\frac{3}{x-1}dx = \frac{x^2}{2}+2x+3\ln|x-1|+c$$

5. This problem computes a recursive relation for the antiderivative of $\sec^n(\theta)$
   a. Compute the derivative of $\sec^{n-2}(\theta)\tan(\theta)$. Eliminate $\tan(\theta)$ from your answer using a trigonometry identity in order to express the derivative in terms of $\sec(\theta)$ alone. Show all your work

$$\begin{aligned}\frac{d}{dx}\sec^{n-2}x\tan x &= (n-2)\sec^{n-3}x\sec x\tan x\tan x+\sec^{n-2}x\sec^2 x\\ &= (n-2)\sec^{n-2}x\tan^2 x+\sec^n x\\ \text{SOLN:}\quad &= (n-2)\sec^{n-2}x\left(\sec^2 x-1\right)+\sec^n x\\ &= (n-2)\sec^n x-(n-2)\sec^{n-2}x+\sec^n x\\ &= (n-1)\sec^n x-(n-2)\sec^{n-2}x\end{aligned}$$

   b. Write your answer from (a) in the form of an integral identity, and then solve for $\int \sec^n(\theta)\,d\theta$ to find a recursive formula in the form

$$\int \sec^n(\theta)\,d\theta = F(\theta)+A\int \sec^{n-2}(\theta)\,d\theta$$

$$\frac{d}{dx}\sec^{n-2}x\tan x = (n-1)\sec^n x-(n-2)\sec^{n-2}x$$

$$\Leftrightarrow (n-1)\int \sec^n x\,dx-(n-2)\int \sec^{n-2}x\,dx = \sec^{n-2}x\tan x$$

$$\Leftrightarrow (n-1)\int \sec^n x\,dx = (n-2)\int \sec^{n-2}x\,dx+\sec^{n-2}x\tan x$$

$$\Leftrightarrow \int \sec^n x\,dx = \frac{n-2}{n-1}\int \sec^{n-2}x\,dx+\frac{\sec^{n-2}x\tan x}{n-1}$$

   c. Use your recursive formula to compute the antiderivative of $\int \sec^4(\theta)\,d\theta$

$$\int \sec^4 x\,dx = \frac{2}{3}\int \sec^2 x\,dx+\frac{\sec^2 x\tan x}{3} = \frac{2}{3}\tan x+\frac{\sec^2 x\tan x}{3}$$

6. Consider $\int \frac{1}{x^2\sqrt{x^2-1}}dx$
   a. Use a trigonometric substitution of the form $x = f(\theta)$ to express the following indefinite integral in terms of the variable θ (assume $x > 1$).

   SOLN: Substitute $x=\sec\theta$ so that $\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec^2\theta\tan\theta} = \int \cos\theta d\theta = \sin\theta+c$

   b. Back substitute to express the integral in terms of x.

   SOLN: $\sin\theta+c = \frac{\sqrt{x^2-1}}{x}+c$

7. In each of the following, use completing the square and a trigonometric substitution to evaluate the indefinite integral.

a. $\int \frac{1}{\sqrt{5x-x^2}} dx$ Assume that $5x - x^2 \geq 0$

SOLN: $\int \frac{dx}{\sqrt{5x-x^2}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x^2-5x+\frac{25}{4}\right)}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x-\frac{5}{2}\right)^2}}$ so substitute

$\left(x-\frac{5}{2}\right)^2 = \frac{25}{4}\sin^2\theta \Leftarrow x = \frac{5}{2}+\frac{5}{2}\sin\theta \Rightarrow dx = \frac{5}{2}\cos\theta d\theta$ and the integral becomes

$\int \frac{dx}{\sqrt{5x-x^2}} = \int d\theta = \theta + c = \arcsin\left(\frac{2}{5}x-1\right)+c$

b. $\int \frac{1}{x^2-6x+18} dx$

SOLN: $\int \frac{dx}{(x-3)^2+9}$ Substitute $(x-3)^2 = 9\tan^2\theta \Leftarrow x = 3+3\tan\theta \Rightarrow dx = 3\sec^2\theta d\theta$ so

the integral is $\int \frac{dx}{(x-3)^2+9} = \int \frac{3\sec^2\theta d\theta}{9\left(\tan^2\theta+1\right)} = \frac{1}{3}\int d\theta = \frac{\theta}{3}+c = \frac{1}{3}\arctan\left(\frac{x-3}{3}\right)+c$

8. Suppose the volume created by rotating the curve $y = e^x$ (where $0 \leq x \leq \ln 2$) about the y-axis is filled with a fluid whose force density is 1000 N/m^3. Find the work required to pump the fluid out the top.
SOLN: Take a horizontal cross-section (at a gravitational equipotential) at height y. This will be a circle with radius $x = \ln y$. The infinitesimal element of volume there is $dV = \pi r^2 dy = \pi(\ln y)^2 dy$ thus the infinitesimal element of weight is $dF = 1000dV = 1000\pi(\ln y)^2 dy$. This weight must be moved to the top of the container, a distance $= 2 - y$. Thus the infinitesimal element of work is $dW = 1000(2-y)dV = 1000\pi(2-y)(\ln y)^2 dy$ and the work required to empty the container is (Note that there are many integration by parts maneuvers in here and that the tables are not shown. To follow this, produce the tables, one by one, favoring $u = \ln y$ or $\ln^2 y$)

$$\int dW = 1000\pi\int_1^2 (2-y)(\ln y)^2 dy = 1000\pi\left(2\int_1^2 (\ln y)^2 dy - \int_1^2 y(\ln y)^2 dy\right)$$

$$= 1000\pi\left(2y(\ln y)^2\Big|_1^2 - 4\int_1^2 \ln y dy - \frac{y^2}{2}(\ln y)^2\Big|_1^2 + \int_1^2 y\ln y dy\right)$$

$$= 1000\pi\left[4\ln^2 2 - 4(y\ln y - y)\Big|_1^2 - 2\ln^2 2 + \frac{y^2}{2}\ln y\Big|_1^2 - \int_1^2 \frac{y}{2}dy\right]$$

$$= 1000\pi\left[4(\ln 2)^2 - 4(2\ln 2 - 1) - 2\ln^2 2 + 2\ln 2 - \frac{y^2}{4}\Big|_1^2\right]$$

$$= 1000\pi\left[2(\ln 2)^2 - 6\ln 2 + 4 - 1 + \frac{1}{4}\right]$$

$$= 1000\pi\left[2(\ln 2)^2 - 6\ln 2 + \frac{13}{4}\right]$$