Math 1B- Project 2 – Spring ’10 (draft)

Some Divergent Trigonometric Integrals

Some integral tables include divergent trigonometric integrals. How do these end up in respectable tables? Historically, it turns out these integrals were originally “evaluated” when some convergent integrals were differentiated under the integral sign with respect to a parameter. We will prove that these integral diverge, look at the history in print and make some observations about necessary and sufficient conditions for differentiating under the integral sign.

Four Divergent Integrals. Here you go! Throughout, *a* and *b* are positive real numbers. Purported values appear on the right:

$$\int_0^\infty x\begin{Bmatrix}\sin\left(ax^2\right)\\ \cos\left(ax^2\right)\end{Bmatrix}\sin(bx)\,dx=\frac{b}{4a}\sqrt{\frac{\pi}{2a}}\left[\sin\left(\frac{b^2}{4a}\right)\pm\cos\left(\frac{b^2}{4a}\right)\right] \tag{1}$$

$$\int_0^\infty x\begin{Bmatrix}\sin\left(ax^2\right)\\ \cos\left(ax^2\right)\end{Bmatrix}\cos(bx)\,dx=$$

$$\frac{1}{2a}\begin{Bmatrix}1\\0\end{Bmatrix}\mp\frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left[\begin{Bmatrix}\sin\left[b^2/(4a)\right]\\ \cos\left[b^2/(4a)\right]\end{Bmatrix}C\left(\frac{b^2}{4a}\right)\mp\begin{Bmatrix}\cos\left[b^2/(4a)\right]\\ \sin\left[b^2/(4a)\right]\end{Bmatrix}S\left(\frac{b^2}{4a}\right)\right] \tag{2}$$

where *C* and *S* are the Fresnel functions,

$$C(x)=\frac{1}{\sqrt{2\pi}}\int_0^x\frac{\cos t}{\sqrt{t}}\,dt \text{ and } S(x)=\frac{1}{\sqrt{2\pi}}\int_0^x\frac{\sin t}{\sqrt{t}}\,dt \tag{3}$$

Note that these are a variation of the Fresnel integrals as defined in Stewart.

Let’s prove that (1) and (2) are divergent.

Proposition. The integrals in (1) and (2) diverge.

Proof. Consider $A:=\int_{-\infty}^{\infty}x\cos\left(x^2+x\right)dx+i\int_{-\infty}^{\infty}x\sin\left(x^2+x\right)dx$ where i is the imaginary unit.

Using the Euler formula $e^{i\theta}=\cos\theta+i\sin\theta$ these integrals can be combined as a single integral:

$A:=\int_{-\infty}^{\infty}xe^{i\left(x^2+x\right)}dx$ where the imaginary unit can be treated like a constant. Since the integrands are continuous, these integrals exist if and only if the limits

$$\lim_{T\to\infty}\int_0^T xe^{i\left(x^2+x\right)}dx \quad \text{and} \quad \lim_{T\to\infty}\int_{-T}^0 xe^{u\left(x^2+x\right)}dx$$

both exist.

Exercise 1: Let T_1, $T_2 > 0$. Integrate by parts and complete the square to show that

$$\int_{-T_1}^{T_2} xe^{i\left(x^2+x\right)}dx = \frac{1}{2i}\left[e^{i\left(T_2^2+T_2\right)} - e^{i\left(T_1^2-T_1\right)}\right] - \frac{e^{-i/4}}{2}\int_{-T_1+1/2}^{T_2+1/2} xe^{ix^2}dx \qquad (4)$$

Exercise 2: Now consider the convergence of $I = \int_{-\infty}^{\infty} e^{ix^2}dx$. Use the substitution $t = x_2$ and the Euler formula to show that $I = \int_0^{\infty}\frac{e^{it}}{\sqrt{t}}dt = \sqrt{2\pi}\left(C(\infty)+iS(\infty)\right)$ and show that this converges.

Exercise 3: Show that as T_1, $T_2 \to \infty$ (independently) the final integral in (4) becomes I but the bracketed term fails to have a limit and so the integral A diverges.

Exercise 4: To get the integrals in (1) and (2) we do the following. Suppose $B = \int_{-\infty}^{\infty} xe^{i\left(x^2-x\right)}dx$ converges. Substitute $u = -x$ to show that $B = -A$ and so B diverges. Next substitute $v = x/\sqrt{a} \pm \left(\sqrt{a}-b\right)/(2a)$ to show that $C = \int_{-\infty}^{\infty} xe^{i\left(ax^2\pm bx\right)}dx$ diverges for all positive values of a and b.

Exercise 5: Finally, explain how if the integrals in (1) and (2) converge then we can form the four convergent linear combinations

$$\int_0^{\infty} x\left[\cos\left(ax^2\right)\cos(bx) \mp \sin\left(ax^2\right)\sin(bx)\right]dx$$
$$\int_0^{\infty} x\left[\sin\left(ax^2\right)\cos(bx) \pm \cos\left(ax^2\right)\sin(bx)\right]dx$$

And how, using addition formulas for the sine and cosine functions followed by Euler's formula, these yield C. Hence, the integrals in (1) and (2) must diverge.

Exercise 6: To see the manner in which the integrals diverge, let

$$A_T = \int_{-T}^{T} xe^{i\left(x^2+x\right)}dx = e^{iT^2}\sin T - \frac{e^{-i/4}}{2}\int_{-T+1/2}^{T+1/2} e^{ix^2}dx \qquad (5).$$

What happens to the integral as as $T \to \infty$? What happens to $e^{iT^2}\sin T$? What are the implications of that?

Exercise 7: Investigate various CAS such as Maple and Mathematica and how well these software evaluate these types of integrals for both arbitrary *a* and *b* and for specific numerical values. How does $\int_0^\infty \sin\left(3.1x^2\right)\cos\left(2.2x\right)dx$ come out, for instance?

Differentiating the convergent integrals

$$\int_0^\infty x\begin{Bmatrix}\sin\left(ax^2\right)\\ \cos\left(ax^2\right)\end{Bmatrix}\cos\left(bx\right)dx=\frac{1}{2}\sqrt{\frac{\pi}{2a}}\left[\cos\left(\frac{b^2}{4a}\right)\mp\sin\left(\frac{b^2}{4a}\right)\right] \tag{6}$$

$$\int_0^\infty x\begin{Bmatrix}\sin\left(ax^2\right)\\ \cos\left(ax^2\right)\end{Bmatrix}\sin\left(bx\right)dx=$$

$$\sqrt{\frac{\pi}{2a}}\left[\begin{Bmatrix}\cos\left[b^2/\left(4a\right)\right]\\ \sin\left[b^2/\left(4a\right)\right]\end{Bmatrix}C\left(\frac{b^2}{4a}\right)\mp\begin{Bmatrix}\sin\left[b^2/\left(4a\right)\right]\\ \cos\left[b^2/\left(4a\right)\right]\end{Bmatrix}S\left(\frac{b^2}{4a}\right)\right] \tag{7}$$

Under the integral sign with respect to *b* yields the divergent integrals (1) and (2). This doesn't mean the functions defined by (6) and (7) aren't differentiable; it just means we cannot obtain their derivatives by differentiating under the integral. Could we have predicted this in advance? This is a difficult problem.