Math 1A – Calculus – Chapter 4 Take-home problems Name__________________

Show your work for credit. Do not copy other peoples work. Describe calculator use explicitly. What buttons did you push to get your results?

1. What calculator(s) are you using to solve the problems on this paper?

2. Suppose a particle is moving along the curve $y = x \cdot e^{-x^4/8}$ so that $\frac{dx}{dt} = 2$. Find $\frac{dy}{dt}$ when $x = 3$. Approximate to four significant digits.

3. Find the coordinates of the inflection points for $y = e^{x - x^2/8}$ accurate to four significant digits

4. Estimate the left-most coordinates on the curve $x = t^4 - 2t^2, y = t + \ln t$. Can you find exact values for these coordinates?

5. Use a graph to estimate the value of the limit. Then use L'Hospital's rule to find the exact value value: $\lim_{x \to \pi/4} (\tan x)^{\sec(2x)}$.

6. Find the coordinates of the point on curve $y = \arctan(x)$ closest to the point (0,2).

7. Find the value(s) of x on the interval [0,2] that satisfy the conclusions of the mean value theorem for $y = x(x^3 - x - 1)$. Approximate to 4 significant digits.

8. Investigate Newton's method for the family of cubic polynomials, $f(x) = (x+2)(x^2 + c)$
   a. Can you find a two-cycle in the case where $c = -1$?
   b. For values of c between 0 and 0.2, what dynamics do you observe from the initial value $x = -\frac{2}{3}$?

Math 1A – Calculus – Chapter 4 Take-home problems Solutions

1. What calculator(s) are you using to solve the problems on this paper?
   SOLN: I'll be using a variety including TI82, TI83, TI85, TI86, TI89, TI92. Should have an HP….

2. Suppose a particle is moving along the curve $y = x \cdot e^{-x^4/8}$ so that $\frac{dx}{dt} = 2$. Find $\frac{dy}{dt}$ when $x = 3$. Approximate to four significant digits.

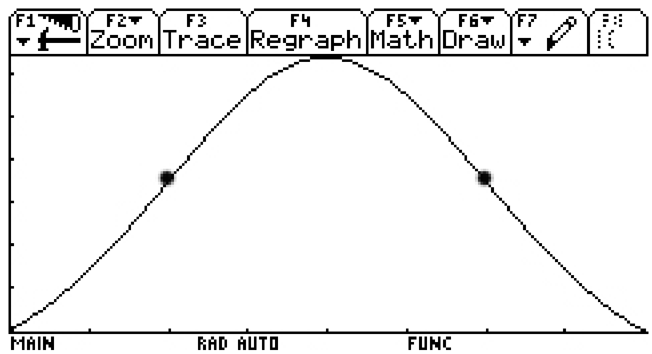
   SOLN: $y = x \cdot e^{-x^4/8} \Rightarrow \frac{dy}{dx} = e^{-x^4/8} - \frac{x^4}{2} e^{-x^4/8} = \frac{e^{-x^4/8}}{2}\left(2 - x^4\right)$

   so that $\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = \frac{dx}{dt}\frac{e^{-x^4/8}}{2}\left(2 - x^4\right) = e^{-x^4/8}\left(2 - x^4\right)$. When $x = 3$,

   $\frac{dy}{dt} = e^{-81/8}\left(2 - 81\right) = -79e^{-10.125}$. So who needs a calculator? Well, you can approximate this using a TI82 to get dy/dt approximately –0.0031651585

   ```
   -79*e^-10.125
        -.0031651585
   ```

3. Find the coordinates of the inflection points for $y = e^{x - x^2/8}$ accurate to four significant digits

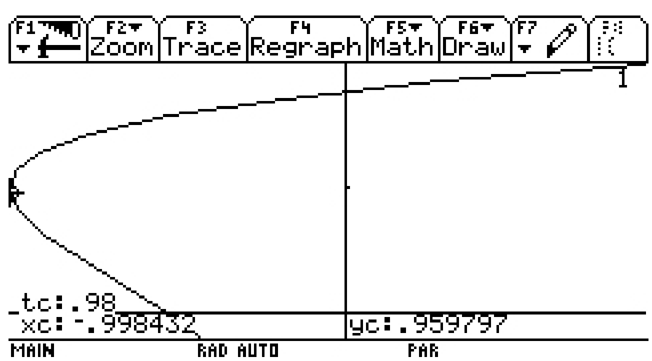
   $$\frac{dy}{dx} = e^{x - x^2/8} = \left(1 - \frac{x}{4}\right)e^{x - x^2/8} \Rightarrow$$

   SOLN: $$\frac{d^2y}{dx^2} = \left(1 - \frac{x}{4}\right)^2 e^{x - x^2/8} - \frac{1}{4}e^{x - x^2/8}$$

   $$= \left(\left(\frac{x}{4} - 1\right)^2 - \frac{1}{4}\right)e^{x - x^2/8}$$

   changes sign where $\left(\frac{x}{4} - 1\right)^2 = \frac{1}{4} \Leftrightarrow x = 4 \pm 2$. Thus the coordinates of the inflection points are $\left(2, e^{3/2}\right) = \left(2, \sqrt{e^3}\right), \left(6, e^{3/2}\right) = \left(6, \sqrt{e^3}\right)$. Again, who needs a calculator? Oh, to estimate accurate to four significant digits, a calculator is handy. The TI82 yields `e^(3/2)  4.48168907` so the points are (2,4.482) and (6,4.482). To be sure, here's a graph as displayed on the TI92+: (baseline along y=1.)

4. Estimate the left-most coordinates on the curve $x = t^4 - 2t^2, y = t + \ln t$. Can you find exact values for these coordinates?

   SOLN: At the left-most coordinates we'll have $\frac{dx}{dy} = 0 \Leftrightarrow \frac{dx/dt}{dy/dt} = \frac{4t^3 - 4t}{1 + 1/t} = \frac{4\left(t^4 - t^2\right)}{t+1}$

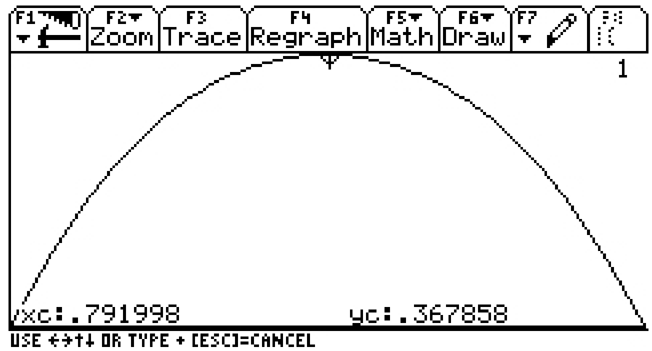
   $$= 4t^2\left(t - 1\right) = 0$$

   and

since $t = 0$ is not in the domain of y, it must be where $t = 1$ so that $(x, y) = (-1,1)$. Well, darn…who needs these stinking calculators? They're good for a graph. Below we see the trace is nearly to the far left point, which corroborates our previous result convincingly:

5. Use a graph to estimate the value of the limit. Then use L'Hospital's rule to find the exact value value: $\lim_{x\to\pi/4}\left(\tan x\right)^{\sec(2x)}$.

   SOLN: Graphing the function on the interval $(0,\pi/2)$ and tracing (it starts in the middle) we see the limit appears to be a local max near (0.792, 0.368)



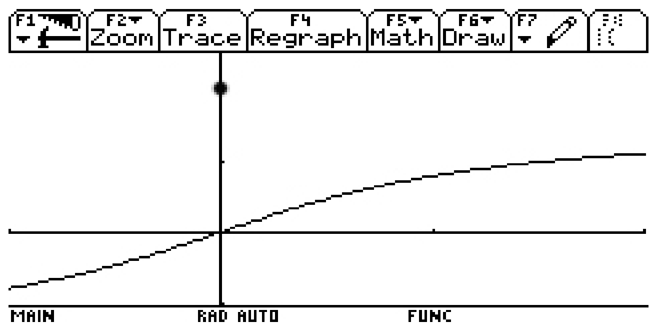
   This is a 1^∞ situation, so we look at

$$\lim_{x\to\pi/4}\ln y=\lim_{x\to\pi/4}\sec(2x)\ln(\tan x)=\lim_{x\to\pi/4}\frac{\ln(\tan x)}{\cos(2x)}=\lim_{x\to\pi/4}\frac{\frac{\sec^2 x}{\tan x}}{-2\sin(2x)}=\lim_{x\to\pi/4}\frac{1}{-2\sin(2x)\sin x\cos x}$$

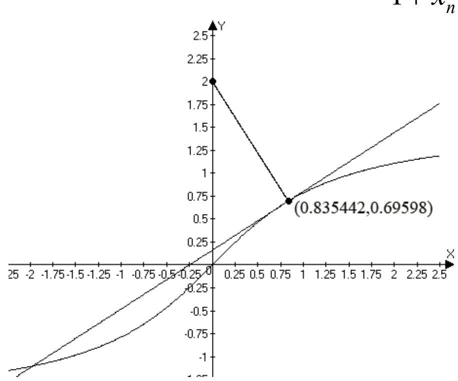
$=\lim_{x\to\pi/4}\frac{1}{-\sin^2(2x)}=-1$, which means that $y=\frac{1}{e}\approx 0.36788$, sure enough!

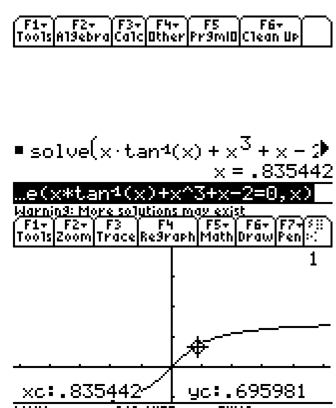
6. Find the coordinates of the point on curve $y=\arctan(x)$ closest to the point (0,2).

   SOLN: The situation is depicted at right. Evidently the point nearest (0,2) is where the normal to the curve passes through (0,2). The slope of the tangent at $x=a$ is $y=\left(1+a^2\right)^{-1}$ so the slope of the normal is $y=-\left(1+a^2\right)$ so the equation of the normal line through (0,2) and (a, $a$arctan(a)) is $y=2-\left(1+a^2\right)x$. Requiring that the normal actually passes through the point of normalcy, $a\arctan(a)=2-\left(1+a^2\right)a=-a^3-a+2$, which is true iff $a\arctan(a)+a^3+a-2=0$. This is an equation not easily solvable by hand. We could try Newton's method. You'd iterate $x_{n+1}=x_n-\frac{x_n\arctan(x_n)+x_n^3+x_n-2}{\arctan(x_n)+\frac{x_n}{1+x_n^2}+3x_n^2+1}$. Yikes. You



   might find it more user-friendly to use the "solve" feature on the TI89 as shown at right, where we see $x\approx 0.835442$ with an ominous warning about more solutions possibly existing (obviously there aren't more.) Using zoom square gives a more reasonable picture that this is, in fact, where the nearest point to (0,2) on the curve is.





7. Find the value(s) of x on the interval [0,2] that satisfy the conclusions of the mean value theorem for $y = x\left(x^3 - x - 1\right)$. Approximate to 4 significant digits.

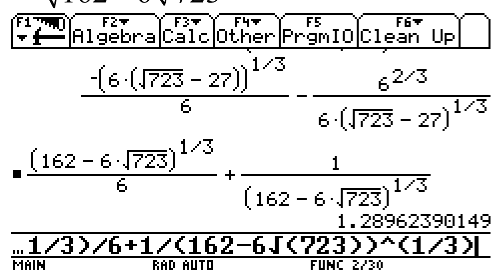
SOLN: Solve for x: $f'(x) = \dfrac{f(2) - f(0)}{2-0} \Leftrightarrow 4x^3 - 2x - 1 = 5 \Leftrightarrow x^3 - \dfrac{1}{2}x = \dfrac{3}{2}$. Using Tartaglia's method we note that, for all a and b, $(a-b)^3 + 3ab(a-b) = a^3 - b^3$ so let $x = a - b$ and set

$3ab = -\dfrac{1}{2} \Leftrightarrow b = -\dfrac{1}{6a}$ and substitute into $a^3 - b^3 = \dfrac{3}{2} \Leftrightarrow a^3 - \left(-\dfrac{1}{6a}\right)^3 = \dfrac{3}{2} \Leftrightarrow a^6 - \dfrac{3}{2}a^3 = -\dfrac{1}{216}$ to

which we add $\left(\dfrac{3}{4}\right)^2$ to complete the square and get $\left(a^3 - \dfrac{3}{4}\right)^2 = -\dfrac{1}{216} + \dfrac{9}{16} = \dfrac{241}{432} = \dfrac{723}{36^2}$ whence

$a^3 - \dfrac{3}{4} = \dfrac{-\sqrt{723}}{36} \Leftrightarrow a = \dfrac{1}{6}\sqrt[3]{162 - 6\sqrt{723}}$ This means that $b = -\dfrac{1}{\sqrt[3]{162 - 6\sqrt{723}}}$

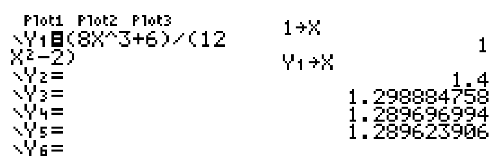
whence $x = a - b = \sqrt[3]{162 - 6\sqrt{723}} + \dfrac{1}{\sqrt[3]{162 - 6\sqrt{723}}}$ and using the TI92+, we see that x is approximately 1.231



Alternatively we could use Newton's method to find a zero of $4x^3 - 2x - 6 = 0$ by iterating

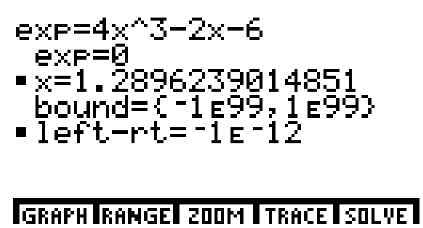
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{4x_n^3 - 2x_n - 6}{12x_n^2 - 2} = \frac{8x_n^3 + 6}{12x_n^2 - 2}$$

To do this on the TI83, enter in the function on the Vars page by hitting the "Y=" button and expressing the formula as shown in the first screen shot at left. Then "quit" and on the "home page" store an initial guess in x using the "STO" key



And then hit the "Vars" button to bring up the menu to find Y1 and place it on the home page by arrowing over/down to highlight and pressing enter. The hit the "Sto" key and store the output of Y1 back to the input as shown in the second screen capture at right. Then just keep hitting enter. As you can see, Newton's method then converges rapidly to the same value we got by the more intricate method above.

Finally, if you like the "Kraft Cheese" method, you could use the "solve" feature on, say, the TI85. On the 85 you hit "2nd+GRAPH" to access the solver menu and then type in the formula in the "exp" field and set its value to zero as shown in the screen capture at right. Then put an initial gues in the field for x and, with the cursor still in that field, hit F5 (SOLVE). The result gives you two more digits than the TI92+. GO 85!



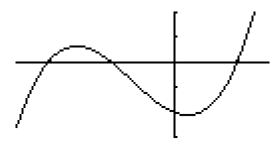
8. Investigate Newton's method for the family of cubic polynomials, $f(x)=(x+2)(x^2+c)$

a. Can you find a two-cycle in the case where $c=-1$?

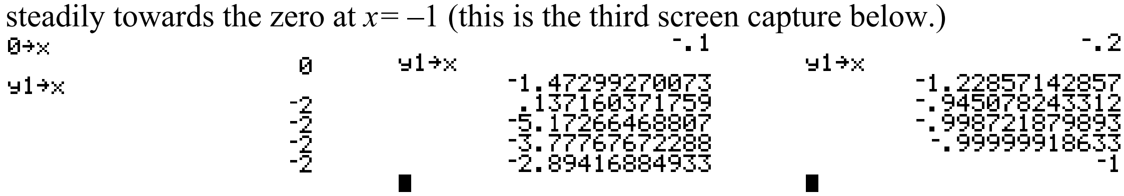
SOLN: Newton's method iterates $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}=x_n-\dfrac{x_n^3+2x_n^2+cx_n+2c}{3x_n^2+4x_n+c}=\dfrac{2(x_n^3+x_n^2-c)}{3x_n^2+4x_n+c}$

Ok, haven't used the '86 yet, so on the '86 enter the iteration formula as y_1 (first screen capture below.) The store -1 into C using the "STO" button. You can then start experimenting with various initial values. If you start with $x_1=1$, then you'll stay there (a "1 cycle.") Starting at $x_1=0.5$ first pushes the iterates beyond 1 and then they decrease, approaching 1 from above (3rd screen shot below.) To get some idea of the dynamics of this process, look at a graph of the function whose zeros we seek. By inspection, we see that the initial value will be somewhere between the local extrema

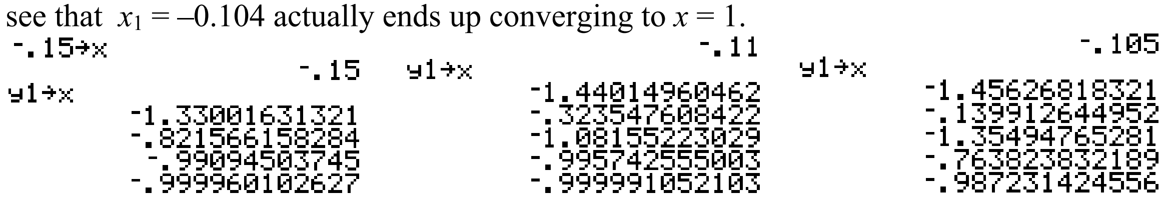
$$f'(x)=\frac{d}{dx}\left(x^3+2x^2-x-2\right)=3x^2+4x-1=0 \Leftrightarrow x=\frac{-2\pm\sqrt{7}}{3}\approx 0.1076 \text{ and } -1.549$$

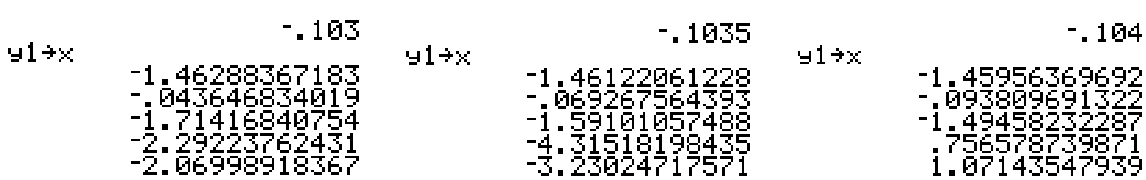


So we start experimenting with iterating Newton's formula with various initial values. Below we see $x_1=0$ immediately goes to the zero at –2 (first screen capture below.) Nudging the initial value a bit to the left, $x_1=-0.1$ we get iterates that initially oscillate but then settle in a path increasing steadily towards –2 (second screen capture below.) Nudging a notch further to the left, $x_1=-0.2$, the next iterate ends up to the right of the local max and then advances steadily towards the zero at $x=-1$ (this is the third screen capture below.)

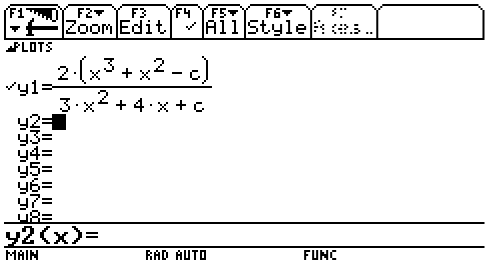


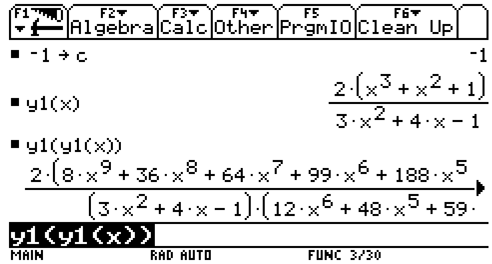
Experimenting with various values between –0.1 and –0.2, we get various results converging towards either –2 or –1, but as we zero in on what seems to lead to a two-cycle we see that $x_1=-0.104$ actually ends up converging to $x=1$.

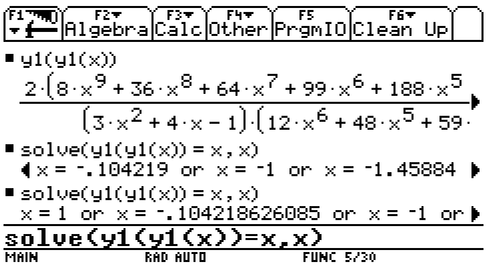


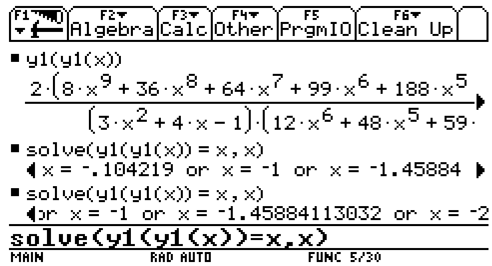


Looks we could use some heavier guns. This could be a job for the TI92+! So for a two-cycle we want $N(N(x))=x$. On the TI92+ we can enter the formula for the Newton's method iteration and put –1 in c as we did on the TI86, but with the extra computer algebra features of the TI92+, we can actually use the calculator to approximate solutions to $N(N(x))=x$. This is shown in the sequence of TI92+ screen captures which follow.









Evidently, either $x_1 = -0.104218626085$ or $x_1 = -1.45884113032$ leads to the two-cycle oscillating from the first back to the second or vice-versa. Let's go back to the TI86 and see if this actually works. Well, as the screen captures below show, it almost works: the trouble is the two-cycle is unstable, so any slight deviation from the perfect initial value will eventually wobble off the cycle, as shown in the sequence of values shown below. Initially it shows nice oscillatory behavior, but the errors compound and it ends up attracting to -1. So there is a two-cycle, but it's unstable and since our initial value is necessarily an approximation, it will eventually wobble away.

```
-.104218626085→x
        -.104218626085
y1→x
        -1.45884113032
        -.104218626104
        -1.45884113026
        -.104218626976

-1.45884112738
-.10421866815
-1.45884099141
-.104220610531
-1.45883457701
-.104312235439
-1.45853210512

-.108617464235
-1.44454751672
-.279342353503
-1.12170087593
-.989243938329
-.999943990688
-.999999998432
```

b. For values of c between 0 and 0.2, what dynamics do you observe from the initial value $x = -\dfrac{2}{3}$?

SOLN: This is an open ended question and an invitation to experiment.

Let's start with $c = 0.1$:

```
.1→C
                    .1
-2/3→x
        -.666666666667
y1→x
        -.078078078078
         .972868430152

 .972868430152
 .51743359262
 .206040617181
-.092818481367
 .751212926103
 .370271099404
 .088200869657

 .088200869657
-.384484136709
 .018119498534
-1.14912998972
 1.109961087
 .606978054533
 .27086284786

 .27086284786
-.009634145132
-3.2363152556 3
-2.53258420159
-2.1559677615 3
-2.01935017815
-2.00035531835
```

Here we see quite a bit of roaming about before settling in on the zero at –2.

How about $c = 0.15$:

```
.15→C
                   .15
-2/3→x
        -.666666666667
y1→x
         .003129890454
        -1.84547698034

-1.84547698034
-2.02951994197
-2.0008059556
-2.0000062537
-2
-2
-2
```

With $c = 0.5$

```
.5→C
                    .5
-2/3→x
        -.666666666667
y1→x
         .844444444444
         .270979979117

 .270979979117
-.450803031432
 1.12001965564
 .493959638896
-.08446817136
-5.37745989295
-3.86615935603

-3.86615935603
-2.90130394607
-2.33318599852
-2.06908390838
-2.0038977316 4
-2.00001343735
-2.00000000016
```

With $c = 0.9$

.9→C	2.07942989333	1.68918315302	-2.56508568918
.9	1.11903127003	.835327962894	-2.15784091215
-2/3→x	.384003405119	.120178108501	-2.01723773553
-.666666666667	-.483545943154	-1.24128681822	-2.00023791453
y1→x	4.68389192365	-4.56461184543	-2.00000004619
3.47008547009	2.89748334024	-3.3299425527	-2
2.07942989333	1.68918315302	-2.56508568918	-2

So we experiment with various values until we discover this interesting zone close to $c = 0.2$ where we find an attractive 2-cycle with $c = 0.196$:

.196→C	-.680073772666	-.678327377824	-.678373128027
.196	.084507850707	.084419626584	.084421778291
-2/3→x	-.677838959974	-.678385022813	-.678371697036
-.666666666667	.084397189132	.084422339126	.084421710859
y1→x	-.678524003805	-.678368223786	-.678372114642
.084147453432	.084428934948	.084421547227	.084421730537
-.680073772666	-.678327377824	-.678373128027	-.678371992778

-.678371992778	-.678372020991	-.67837202029	-.678372020307
.084421724795	.084421726124	.084421726091	.084421726092
-.67837202834	-.678372020107	-.678372020312	-.678372020307
.08442172647	.084421726083	.084421726092	.084421726092
-.678372017962	-.678372020365	-.678372020305	-.678372020307
.084421725981	.084421726095	.084421726092	.084421726092
-.678372020991	-.67837202029	-.678372020307	-.678372020307

An attractive 4-cycle with $c = 0.19$:

.19→C	-.738586601788	-.666225694837	-.738583874435
.19	.084049769196	.073210613837	.084048910772
-2/3→x	-.666220267135	-.738583876057	-.666225566829
-.666666666667	.073210623181	.084048911282	.073210614056
y1→x	-.738583808558	-.666225563679	-.738583874474
.073210236476	.084048890037	.073210614061	.084048910784
-.738586601788	-.666225694837	-.738583874435	-.666225566753

-.666225566753	-.738583874473
.073210614056	.084048910784
-.738583874473	-.666225566755
.084048910784	.073210614056
-.666225566755	-.738583874473
.073210614056	.084048910784
-.738583874473	-.666225566755

And, zounds! An attractive 8-cycle with $c = 0.188$

.188→C	-.760344099852	-.667710529157	-.746611524886
.188	.088380820329	.069592065864	.083032575903
-2/3→x	-.635441422665	-.760328064095	-.667635408324
-.666666666667	.071421790468	.08837398491	.069591773897
y1→x	-.746577471574	-.635481098593	-.760330289964
.069589962489	.083020545359	.071417193922	.088374933627
-.760344099852	-.667710529157	-.746611524886	-.635475591584

-.635475591584	-.760329991065	-.667644431627	-.746607349494
.071417831584	.088374806228	.069591807814	.083031100476
-.74660680065	-.635476331093	-.760330031391	-.667644620469
.083030906542	.071417745949	.088374823416	.069591808527
-.667645831353	-.746607435088	-.635476231322	-.760330025954
.069591813103	.083031130721	.071417757503	.088374821099
-.760329991065	-.667644431627	-.746607349494	-.635476244775

-.635476244775	-.760330026687	-.66764459844	-.746607359688
.071417755945	.088374821411	.069591808444	.083031104078
-.746607361035	-.635476242961	-.760330026588	-.667644597977
.083031104554	.071417756155	.088374821369	.069591808442
-.667644595007	-.746607359479	-.635476243205	-.760330026602
.069591808431	.083031104004	.071417756127	.088374821375
-.760330026687	-.66764459844	-.746607359688	-.635476243173

-.635476243173	-.7603300266	-.667644598031	-.746607359663
.07141775613	.088374821374	.069591808442	.083031104069
-.74660735966	-.635476243177	-.7603300266	-.667644598033
.083031104068	.07141775613	.088374821374	.069591808442
-.667644598039	-.746607359664	-.635476243176	-.7603300266
.069591808442	.08303110407	.07141775613	.088374821374
-.7603300266	-.667644598031	-.746607359663	-.635476243176