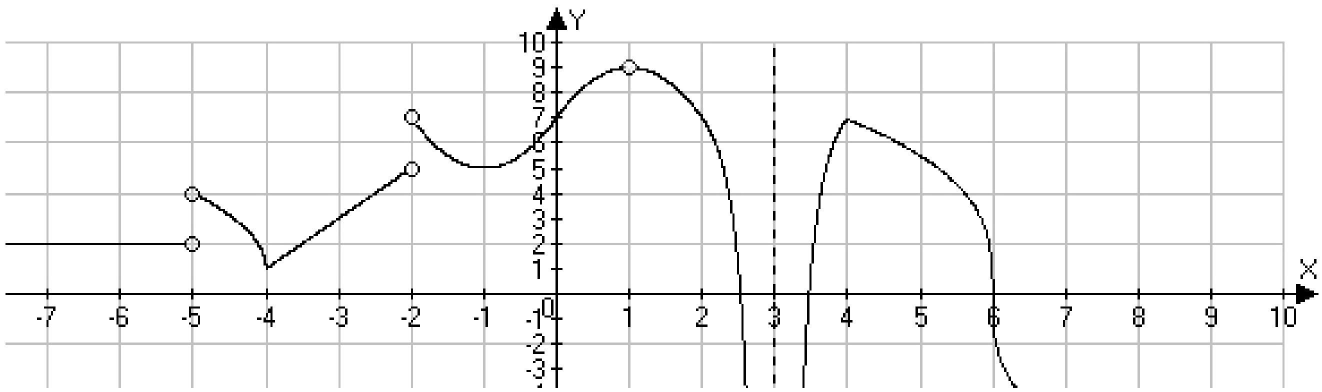
Math 1A – Final Exam – Spring 08 Name___________________

Show your work for credit. Write all responses on separate paper. Do not use a calculator.

1. Consider the following graph for the derivative function $f(x) = F'(x)$. That is, the graph shows the values slopes of the tangent line to $y = F(x)$ for values of x between about –8.7 and 6.7. In what follows, be very careful to distinguish between $F(x), f(x)$ and $f'(x)$.



a. Note that this tells us nothing about the actual values of $F(x)$, just how $F(x)$ is changing, so further assume that $F(-7) = 0$, what would be the value of $F(-6)$?

SOLN: Since F is increasing at a constant rate of 2 per 1 during this interval, $\boxed{F(-6) = 2}$

b. Based on the graph, find $\lim_{x \to -5^-} F(x)$ if it exists, if not, explain why not.

SOLN: Since F continues to grow at this rate throughout the interval up to $x = -5$, $\boxed{\lim_{x \to -5^-} F(x) = 4}$

c. Where does $f(x)$ is have a jump discontinuity? List all values of x where this is true.
SOLN: $f(x)$ has jump discontinuities where $\boldsymbol{x = -5}$ **and** $\boldsymbol{x = -2}$

d. Where does $f(x)$ is have a removable discontinuity? List all values of x where this is true.
SOLN: $f(x)$ has a removable discontinuity where $\boldsymbol{x = 1}$.

e. Where does $f'(x)$ have a jump discontinuity?

SOLN: $f'(x)$ has jump discontinuity wherever there's an abrupt change in slope from one finite value to another. This happens where $\boldsymbol{x = -5, x = -4, x = -2}$ **and** $\boldsymbol{x = 4}$.

f. Over what interval(s) is $F(x)$ increasing?
SOLN: F is increasing where $f(x) > 0$ which is true on
(–7.5, –5), (–5, –2), (–2, 1), (1 , 2.6), (3.6, 6)

g. Where does $F(x)$ have inflection points?
SOLN: $F(x)$ has inflection points where $F''(x) = f'(x)$ changes sign. This is were f is continuous and stops increasing and starts decreasing, or vice versa. Thus F has inflection points where $\boldsymbol{x = -4, x = -1}$, **and** $\boldsymbol{x = 4}$. You may thing there ought to be an inflection point where $x = 1$, but since $\lim_{x \to 1} F'(x) = \lim_{x \to 1} f(x) = 9$ exists while $f(1)$ does not exist, it must be that F has either a jump or removable discontinuity where $x = 1$ and so there is no inflection point.

h. Where is $f(x)$ defined and yet $f'(x)$ is not defined?
SOLN: This will be where f is continuous but has no tangent line of finite slope. That's true where $\boldsymbol{x = -4, x = 4}$, **and** $\boldsymbol{x = 6}$.

2. If the tangent to $y = f(x)$ at (0,2) passes through the point (3,0), find $f'(0)$.

SOLN: Clearly $f'(0) = \frac{2-0}{0-3} = -\frac{2}{3}$

3. Is there a number a such that $\lim_{x\to 1} \frac{x+a}{x^2+x-2}$ exists? If not, why not? If so, find the value of a and the value of the limit.

SOLN: $\lim_{x\to 1} \frac{x+a}{x^2+x-2}$ exists only if $a = -1$: $\lim_{x\to 1} \frac{x-1}{x^2+x-2} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+2)} = \lim_{x\to 1} \frac{1}{x+2} = \frac{1}{3}$

4. Find the limit. Explain your answers.

a. $\lim_{x\to 3^+} \frac{\sqrt{x^2-9}}{x-3} = \lim_{x\to 3^+} \frac{\sqrt{(x-3)(x+3)}}{x-3} = \lim_{x\to 3^+} \frac{\sqrt{x+3}}{\sqrt{x-3}} = \infty$

b. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} (-x) = 0$

c. $\lim_{x\to\infty} \frac{1-\cos x}{x} = 0$ by the squeeze theorem: $0 \le \frac{1-\cos x}{x} \le \frac{2}{x}$

5. Consider the function $f(x) = e^x \cos x$

a. Find a formula for the second derivative $f''(x)$.

SOLN: $f'(x) = e^x(\cos x - \sin x) = -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)e^x$ so that

$f''(x) = e^x(-\sin x - \cos x + \cos x - \sin x) = -2\sin x\, e^x$

b. Over what interval(s) is $f(x)$ concave up?

SOLN: $f(x)$ is concave up where $\sin x < 0$; that is, on intervals of the form $((2k-1)\pi, 2k\pi)$ where k is any integer.

6. Show that $\lim_{u\to 0} \frac{1-\cos u}{u} = 0$ and use the definition of the derivative to find $f'(x)$ where $f(x) = \sin x$. You may assume $\lim_{u\to 0} \frac{\sin u}{u} = 1$.

SOLN:
$$\lim_{u\to 0} \frac{1-\cos u}{u} = \lim_{u\to 0} \frac{(1-\cos u)}{u}\frac{(1+\cos u)}{(1+\cos u)} = \lim_{u\to 0} \frac{1-\cos^2 u}{u(1+\cos u)} = \lim_{u\to 0} \frac{\sin^2 u}{u(1+\cos u)}$$
$$= \lim_{u\to 0} \frac{\sin u}{u} \lim_{u\to 0} \frac{\sin u}{1+\cos u} = 1 \cdot 0 = 0$$

Thus
$$f'(x) \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{\beta\to x} \frac{\sin x\cos h + \cos x\sin h - \sin(x)}{h}$$
$$= \cos(x)\lim_{h\to 0} \frac{\sin h}{h} + \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} = \cos x$$

However, you can skip this limit if you use the sum to product identity like so:

$$f'(x) \equiv \lim_{\beta\to x}\frac{f(x)-f(\beta)}{x-\beta} = \lim_{\beta\to x}\frac{\sin(x)-\sin(\beta)}{x-\beta} = \lim_{\beta\to x}\frac{\cos\left(\frac{x+\beta}{2}\right)\sin\left(\frac{x-\beta}{2}\right)}{\frac{x-\beta}{2}}$$

$$= \lim_{\beta\to x}\cos\left(\frac{x+\beta}{2}\right)\lim_{\frac{x-\beta}{2}\to 0}\frac{\sin\left(\frac{x-\beta}{2}\right)}{\frac{x-\beta}{2}} = \cos x$$

7. Suppose a function $y = f(x)$ satisfies the equation $y^2\cos(\pi x)+x^2y+y^3=1$ in a neighborhood of the point (1,1). Find an equation for the tangent line at (1,1).
SOLN: Differentiateing with respect to x we have
$2yy'\cos(\pi x)-\pi y^2\sin(\pi x)+2xy+x^2y'+3y^2y'=0$ and then plug in $x = y = 1$ to get
$-2y'+2+y'+3y'=2y'+2=0 \Leftrightarrow y'=-1$ so the tangent line is $y=1-(x-1)=2-x$.

8. Use a linear approximation to estimate $\arctan\left(\frac{3}{4}\right)$ by considering the tangent line to $y=\arctan x$ at $x=\frac{\pi}{4}$. Approximate to 3 significant digits.
SOLN:

$$\arctan(x) \approx \arctan(a)+\frac{1}{1+a^2}(x-a) \Rightarrow \arctan\left(\frac{3}{4}\right) \approx \arctan\left(\frac{\pi}{4}\right)+\frac{1}{1+\left(\frac{\pi}{4}\right)^2}\left(\frac{3}{4}-\frac{\pi}{4}\right)$$

$$\approx 0.665774+\frac{1}{1+0.61685}(-0.035398) \approx 0.643881$$

This, of course, was not the sensible question to ask. Rather, we should use the value at $x = 1$ to approximate the value at $x = 3/4$ like so:

$$\arctan\left(\frac{3}{4}\right) \approx \arctan(1)+\frac{1}{1+(1)^2}\left(\frac{3}{4}-1\right) = \frac{\pi}{4}-\frac{1}{8} = \frac{2\pi-1}{8} \approx \frac{5.2832}{8} = 0.6604$$

The best three-digit approximation 0.644, so while the first approximation is better, it requires a calculator, where the second doesn't.

9. A spherical balloon is filling with water at a rate of π cm^3/sec. How fast is the surface area increasing when the radius is 2 cm? Useful formulas might include $V=\frac{4}{3}\pi r^3$ and $A=4\pi r^2$.
SOLN: Given that $\frac{dV}{dt}=\frac{dV}{dr}\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}=\pi \Rightarrow \frac{dr}{dt}=\frac{1}{4r^2}$ and $\frac{dA}{dt}=\frac{dA}{dr}\frac{dr}{dt}=8\pi r\left(\frac{1}{4r^2}\right)=\frac{2\pi}{r}$, when $r = 2$, $\frac{dA}{dt}=\pi$ cm^2/sec.

10. Show there are no values of x in the interval (–1,1) that satisfy the conclusion of the mean value theorem for $f(x)=\frac{1}{x}$. Why does this not contradict the theorem?
SOLN: $f'(x)=\frac{-1}{x^2}=\frac{f(1)-f(-1)}{2}=1 \Leftrightarrow x^2=-1$ has no real solution. This doesn't violate the

theorem because $f(x)=\frac{1}{x}$ is not differentiable on (-1,1).

11. If a resistor of r ohms is connected across a battery of V volts with internal resistance R ohms, then the power in watts in the external resistor is $P(r)=\frac{V^2 r}{(r+R)^2}$. If V and R are constant by r varies, what is the maximum value of the power?

SOLN: $P'(r)=\frac{V^2(r+R)^2-2(r+R)rV^2}{(r+R)^4}=\frac{V^2(r+R)-2rV^2}{(r+R)^3}=0 \Leftrightarrow r+R=2r \Leftrightarrow \boxed{r=R}$

12. Consider the equation $\sin 3x = 1-x^3$

a. Use the intermediate value theorem to prove that this equation has a solution in $\left[0,\frac{\pi}{3}\right]$

SOLN: Let $f(x)=\sin 3x-1+x^3$ then if a is a solution to the equation, $f(a)=0$. Now $f(x)$ is a sum of continuous function so it must also be continuous, in particular continuous on $\left[0,\frac{\pi}{3}\right]$ so that, by the Intermediate Value Theorem, since $f(0)=-1<0$ and $f\left(\frac{\pi}{3}\right)=\left(\frac{\pi}{3}\right)^3-1>0$, there exists c in $\left(0,\frac{\pi}{3}\right)$ such that $f(c)=0 \Leftrightarrow \sin 3c = 1-c^3$

b. Use Newton's method to find the next estimate to the solution starting from $x_1=\frac{\pi}{6}$.

SOLN: $x_2=\frac{\pi}{6}-\frac{\sin\left(\frac{\pi}{2}\right)-1+\left(\frac{\pi}{6}\right)^3}{3\cos\left(\frac{\pi}{2}\right)+3\left(\frac{\pi}{6}\right)^2}=\frac{\pi}{6}-\frac{\left(\frac{\pi}{6}\right)^3}{3\left(\frac{\pi}{6}\right)^2}=\frac{\pi}{6}-\frac{\pi}{18}=\frac{\pi}{9}$

13. Find the antiderivative of $f(x)=\frac{8}{x^2}$ which has $y=x$ as a tangent line.

SOLN: $F(x)=\frac{-8}{x}+c \Rightarrow f(x)=F'(x)=\frac{8}{x^2}$. If $y=x$ is tangent to the curve at $x=a$ then $F(a)=\frac{-8}{a}+c=a$ and $F'(a)=\frac{8}{a^2}=1$. Thus $a^2=8 \Leftrightarrow a=2\sqrt{2}$ so that $F(a)=\frac{-8}{2\sqrt{2}}+c=2\sqrt{2} \Leftrightarrow c=2\sqrt{2}+\frac{4}{\sqrt{2}}=4\sqrt{2}$ and so $\boxed{F(x)=\frac{-8}{x}+4\sqrt{2}}$