Math 1A – Chapter 4 Test Reprise – Spring '08 Name________________________

Show your work for credit. Write all responses on separate paper. No calculators.

1. The volume of a cube is increasing at a rate of 120 meter^3/sec. How fast is the surface area changing when the edge length is 10 meters? *Hint*: a cube has six square faces.

2. Let $f(x)=\begin{cases} x^2\ln x & \text{if} \quad x>0 \\ 0 & \text{if} \quad x=0 \end{cases}$
   a. Show that $f(x)$ satisfies the conditions of the extreme value theorem on $[0,e]$. *Hint*: This is a L'Hospital's rule problem.

   b. Use the "closed interval method" to find all max and min on the interval $[0,e]$ and classify each as either local or absolute.

3. Consider the function $f(t)=e^{2\sin t}$ on the interval $0<t<2\pi$.
   a. Where is f increasing on this interval?

   b. Where is f concave up on this interval? *Hint*: Use the Pythagorean identity to get an equation quadratic in sin(x).

4. Find the point on the ellipse $9x^2+y^2=9$ which is closest to the point (0,1). *Hint*: Use the constraint to express y in terms of x. Then use the distance formula. Since the distance is certainly greater than 1, it suffices to minimize the square of the distance.

5. Consider $f(x)=\sqrt[3]{x}$ on $[-1,1]$.
   a. Explain why f doesn't satisfy the conditions of the mean value theorem on that interval.
   b. Nevertheless, find all values of x which satisfy the conclusion of the mean value theorem on that interval and make a sketch to show what's going on.

6. Suppose $x_1 = 5$ is the initial guess for $\sqrt{30}$ in Newton's method with $f(x)=x^2-30$. What would the next iteration x_2 be? Show that $x_3=\dfrac{241}{44}$.

7. Find the most general antiderivative of $f(x)=1.8x^2$. Which antiderivative passes through the point (1,1)?

Math 1A – Chapter 4 Test Reprise Solutions – Spring ’08

1. The volume of a cube is increasing at a rate of 120 meter^3/sec. How fast is the surface area changing when the edge length is 10 meters? *Hint*: a cube has six square faces.

   SOLN: $V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = 3x^2\frac{dx}{dt} = 120$, so when $x = 10$, $\frac{dx}{dt} = \frac{2}{5}$. The surface area is $A = 6x^2$ so

   $$\left.\frac{dA}{dt}\right|_{x=10} = 12x|_{x=10}\frac{dx}{dt} = 120\left(\frac{2}{5}\right) = 48\frac{\text{m}^2}{\text{sec}}$$

2. Let $f(x) = \begin{cases} x^2 \ln x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

   a. Show that $f(x)$ satisfies the conditions of the extreme value theorem on $[0,e]$. *Hint*: This is a L’Hospital’s rule problem.

      SOLN: Both x^2 and $\ln x$ are continuous on $(0,e]$, so their product will be continuous on that interval. To meet the conditions of the Mean Value Theorem, we need to product to be continuous from the right at $x = 0$. Indeed the sufficient condition is met:

      $$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}} = \lim_{x\to 0^+} \frac{-x^2}{2} = 0 = f(0).$$

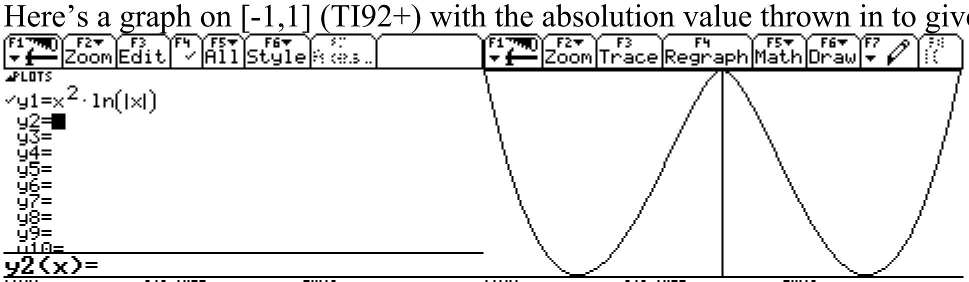
   b. Use the “closed interval method” to find all max and min on the interval $[0,e]$ and classify each as either local or absolute.

      SOLN: $f'(x) = 2x\ln x + x = x(1+2\ln x) = 0 \Rightarrow \ln x = -\frac{1}{2} \Leftrightarrow x = \frac{1}{\sqrt{e}}$ Now $f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e}\left(\frac{-1}{2}\right) = \frac{-1}{2e}$

      is a local min since $f''(x) = 3 + 2\ln x \Rightarrow f''\left(e^{-1/2}\right) = 3 - 1 = 2 > 0$

      Now $f(0) = 0$ is a local max since f is decreasing on $\left(0, \frac{1}{\sqrt{e}}\right)$ and increasing thereafter, (0,0) is a local max, $\left(\frac{1}{\sqrt{e}}, \frac{-1}{e}\right)$ is an absolute min and (e,e^2) is an absolute max.

      Here’s a graph on [-1,1] (TI92+) with the absolution value thrown in to give symmetry:

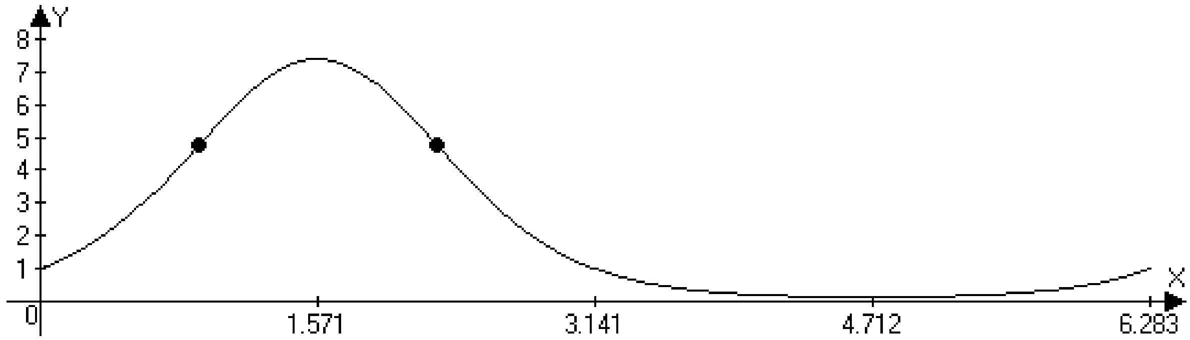


3. Consider the function $f(t) = e^{2\sin t}$ on the interval $0 < t < 2\pi$.

   a. Where is f increasing on this interval?

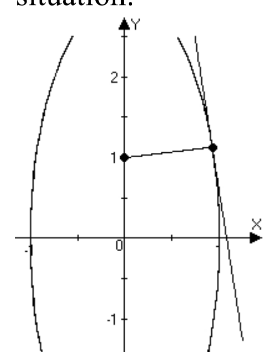
      SOLN: $f'(t) = 2\cos t\, e^{2\sin t} > 0 \Leftrightarrow \cos t > 0$ is true for $t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$.

b. Where is f concave up on this interval? *Hint*: Use the Pythagorean identity to get an equation quadratic in sin(x).
SOLN:
$$f''(t)=\left(4\cos^2 t-2\sin t\right)e^{2\sin t}>0 \Leftrightarrow 4\cos^2 t-2\sin t>0 \Leftrightarrow 4\left(1-\sin^2 t\right)-2\sin t>0$$
$$\Leftrightarrow 2\sin^2 t+\sin t-2>0 \Leftrightarrow 2\sin^2 t+\sin t-2>0 \Leftrightarrow t\in\left(0,\sin^{-1}\frac{\sqrt{17}-1}{4}\right)\cup\left(\pi-\sin^{-1}\frac{\sqrt{17}-1}{4},2\pi\right)$$
Here's what this function looks like. The inflection points are plotted:



4. Find the point on the ellipse $9x^2+y^2=9$ which is closest to the point (0,1). *Hint*: Use the constraint to express y in terms of x. Then use the distance formula. Since the distance is certainly greater than 1, it suffices to minimize the square of the distance.

To be sure, here's an illustration of this situation:



SOLN: $9x^2+y^2=9 \Leftrightarrow y=\pm 3\sqrt{1-x^2}$ Since (0,1) is above the x axis, choose the positive y value in computing the square of the distance:
$D^2=\left(x-0\right)^2+\left(1-3\sqrt{1-x^2}\right)^2$ whence
$$2DD'=2x+2\left(1-3\sqrt{1-x^2}\right)\left(\frac{3x}{\sqrt{1-x^2}}\right)=0 \Leftrightarrow x+\frac{3x}{\sqrt{1-x^2}}-9x=\frac{3x}{\sqrt{1-x^2}}-8x=0$$
To solve, isolate the radical and equate squares:
$$\sqrt{1-x^2}=\frac{3}{8}\Rightarrow 1-x^2=\frac{9}{64}\Leftrightarrow x^2=\frac{55}{64}\Leftrightarrow x=\frac{\pm\sqrt{55}}{8}\approx\pm 0.9270.$$

The y coordinate is then $y=3\sqrt{1-\frac{55}{64}}=\frac{9}{8}$. Note that we can find the slope of the line tangent to the ellipse at that point $18x+2yy'=0 \Leftrightarrow y'=\frac{-9x}{y}=\frac{-9\sqrt{55}/8}{9/8}=-\sqrt{55}$ and that this slope is the negative reciprocal of the slope of the line segment connecting the point to (0,1): $\frac{9/8-1}{\sqrt{55}/8-0}=\frac{1}{\sqrt{55}}$

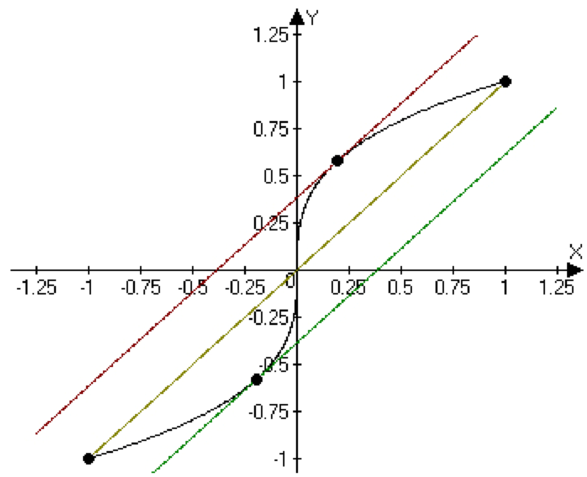
5. Consider $f\left(x\right)=\sqrt[3]{x}$ on [–1,1].

a. Explain why *f* doesn't satisfy the conditions of the mean value theorem on that interval.
SOLN: The function is not differentiable at $x = 0$. There is a vertical tangent line at $x=0$

b. Nevertheless, find all values of x which satisfy the conclusion of the mean value theorem on that interval and make a sketch to show what's going on.
SOLN: Sketch is at right. The points where the slopes of the tangent lines are parallel to the secant line are found by solving

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}} = \frac{\sqrt[3]{1} - \sqrt[3]{-1}}{1-(-1)} = \frac{2}{2} = 1$$

$$\Leftrightarrow \sqrt[3]{x^2} = \frac{1}{3} \Leftrightarrow x = \pm\frac{\sqrt{3}}{9}$$



6. Suppose $x_1 = 5$ is the initial guess for $\sqrt{30}$ in Newton's method with $f(x) = x^2 - 30$. What would the next iteration x_2 be? Show that $x_3 = \frac{241}{44}$.

SON: $x_2 = 5 - \frac{f(5)}{f'(5)} = 5 + \frac{5}{10} = \frac{11}{2}$ so that $x_3 = \frac{11}{2} - \frac{f\left(\frac{11}{2}\right)}{f'\left(\frac{11}{2}\right)} = \frac{11}{2} - \frac{\frac{121}{4} - \frac{120}{4}}{11} = \frac{11}{2} - \frac{1}{44} = \frac{241}{44}$

7. Find the most general antiderivative of $f(x) = 1.8x^2$. Which antiderivative passes through (1,1)?
SOLN: The set of all antiderivatives is found by adding a constant to the most obvious antiderivative: $F(x) = 0.6x^3 + c$. This curve will pass through (1,1) if $c = 0.4$