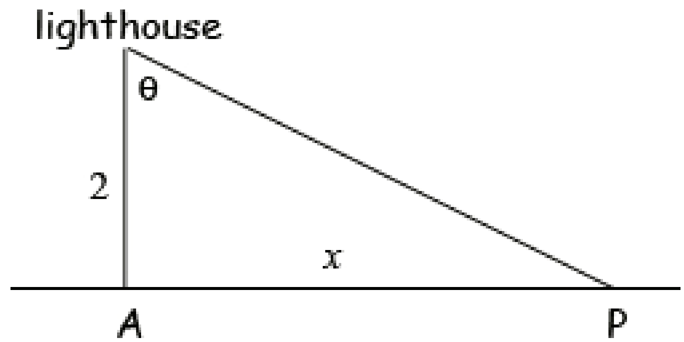
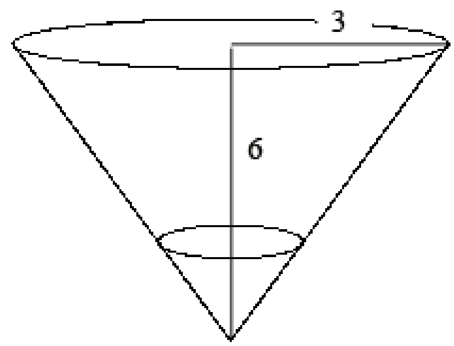
Math 1A – Chapter 4 Test – Spring '08 Name________________________

Show your work for credit. Write all responses on separate paper. No calculators.

1. A lighthouse is located on a mound of rock out in the ocean 2 km from the nearest point *A* on a straight shoreline. If the lamp rotates at a rate of 3 rpm (revolutions per minute), how fast is the lighted spot *P* on the shoreline moving along the shoreline when it is 4 km from point A?



2. A reservoir in the shape of a circular cone is filling with water at a rate of π cubic feet per second. The radius at the top of the cone is 3 feet and the height of the cone is 6 ft. How fast is the depth of water increasing when the depth is 2 ft?



3. Find the critical numbers for $f(x)=x^{3/5}(4-x)$ and find all max and min on the interval [–32 ,32].

4. Let $f'(x)=(x-1)^3(x-3)^4$
   a. On what interval(s) is *f* increasing?
   b. On what interval(s) is *f* concave up?

5. For what values of *a* and *b* does the function $f(x)=\dfrac{ax}{b+x^2}$ have a maximum at $f(1)=2$?

6. Find the limit:
   a. $\lim\limits_{x\to 0}\dfrac{\cos mx-\cos nx}{x^2}$
   b. $\lim\limits_{x\to -\infty} x^3e^{2x}$

7. Find all values of *x* that satisfy the conclusion of the mean value theorem for $f(x)=x^3$ on the interval [0,2].

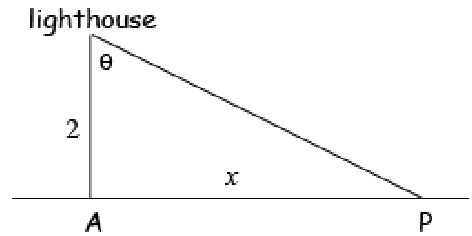
8. Show that the rectangle of largest area that can be inscribed in a circle is a square.

9. Find the area of the largest rectangle that can fit above the *x*-axis and below the curve $y=e^{-x^2}$.

10. Find an inflection point for $f(x)=\ln(2+\sin 3x)$.

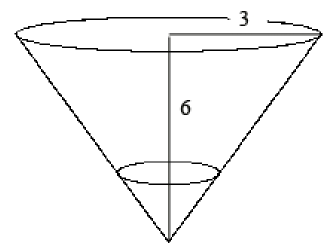
Math 1A – Chapter 4 Test Solutions – Spring '08

1. A lighthouse is located on a mound of rock out in the ocean 2 km from the nearest point *A* on a straight shoreline. If the lamp rotates at a rate of 3 rpm (revolutions per minute), how fast is the lighted spot *P* on the shoreline moving along the shoreline when it is 4 km from point A?



SOLN: We're given $\frac{d\theta}{dt} = \frac{3\text{rev}}{\text{min}} \cdot \frac{2\pi}{\text{rev}} = 6\pi$. Since $\tan\theta = \frac{x}{2}$, multiplying through by 2 and differentiating with respect to *t*, $2\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$. When *x* = 4, $\sec\theta = \frac{\sqrt{2^2+4^2}}{2} = \sqrt{5}$ so that $\frac{dx}{dt} = 2(5)6\pi = 60\pi \frac{\text{km}}{\text{min}}$

2. A reservoir in the shape of a circular cone is filling with water at a rate of π cubic feet per second. The radius at the top of the cone is 3 feet and the height of the cone is 6 ft. How fast is the depth of water increasing when the depth is 2 ft?



SOLN: The ratio of radius to height is constant throughout: $\frac{r}{h} = \frac{1}{2}$ so

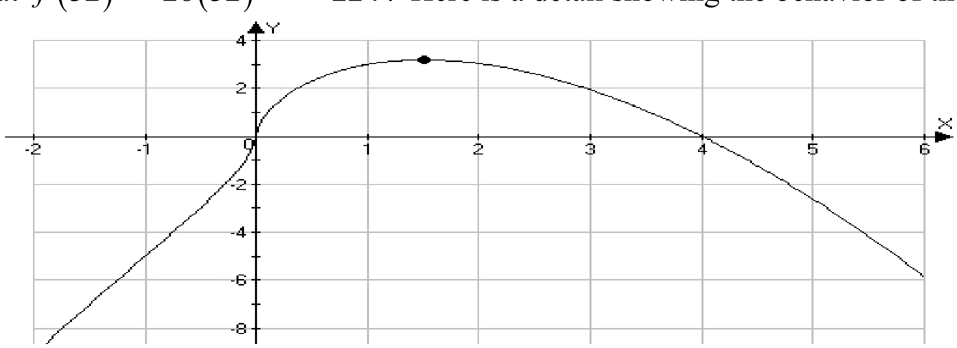
$$V = \frac{1}{3}\pi r^2 h = \frac{2\pi r^3}{3} \Rightarrow \frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt} = \pi \Rightarrow \frac{dr}{dt} = \frac{1}{2r^2}$$

When *h* = 2, *r* = 1 so $\frac{dh}{dt} = 1$ foot per second.

3. Find the critical numbers for $f(x) = x^{3/5}(4-x)$ and find all max and min on the interval [–32 ,32].

SOLN: $f'(x) = \frac{3}{5}x^{-2/5}(4-x) - x^{3/5} = \frac{12-3x}{5x^{2/5}} - \frac{x}{x^{2/5}} = \frac{12-8x}{5x^{2/5}}$ So the critical points are *x* = 0 (where the tangent line is vertical) and *x* = 3/2, where the tangent line is horizontal. Since the derivative function is positive on $(-\infty,0)\cup\left(0,\frac{3}{2}\right)$, so it's actually increasing on $\left(-\infty,\frac{3}{2}\right)$ and decreasing on $\left(\frac{3}{2},\infty\right)$ On the interval [–32 ,32] the function has local max at

$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^{3/5}\left(4-\frac{3}{2}\right) = \frac{5\sqrt[5]{27}}{2\sqrt[5]{8}} = \frac{5\sqrt[5]{108}}{4}$, a global min at $f(-32) = 36(-32)^{3/5} = -288$ and local min at $f(32) = -28(32)^{3/5} = -224$. Here is a detail showing the behavior of the function near the max.



4. Let $f'(x)=(x-1)^3(x-3)^4$

   a. On what interval(s) is *f* increasing?

      SOLN: $f'(x)=(x-1)^3(x-3)^4>0 \Leftrightarrow x>1$ so *x* is increasing on $(1,\infty)$

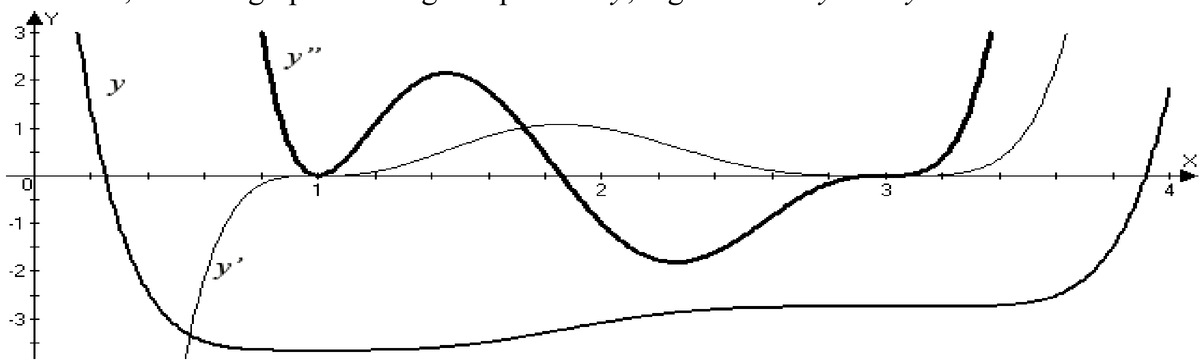
   b. On what interval(s) is *f* concave up?

      SOLN: *f* is concave up where $f''(x)=3(x-1)^2(x-3)^4+4(x-1)^3(x-3)^3$

      $=(x-1)^2(x-3)^3(3(x-3)+4(x-1))=(x-1)^2(x-3)^3(3(x-3)+4(x-1))$

      $=(x-1)^2(x-3)^3(7x-13)>0$ which is true if $x\in\left(-\infty,\frac{13}{7}\right)\cup(3,\infty)$

      To be sure, here's a graph showing one possible *y*, together with *y'* and *y''*:

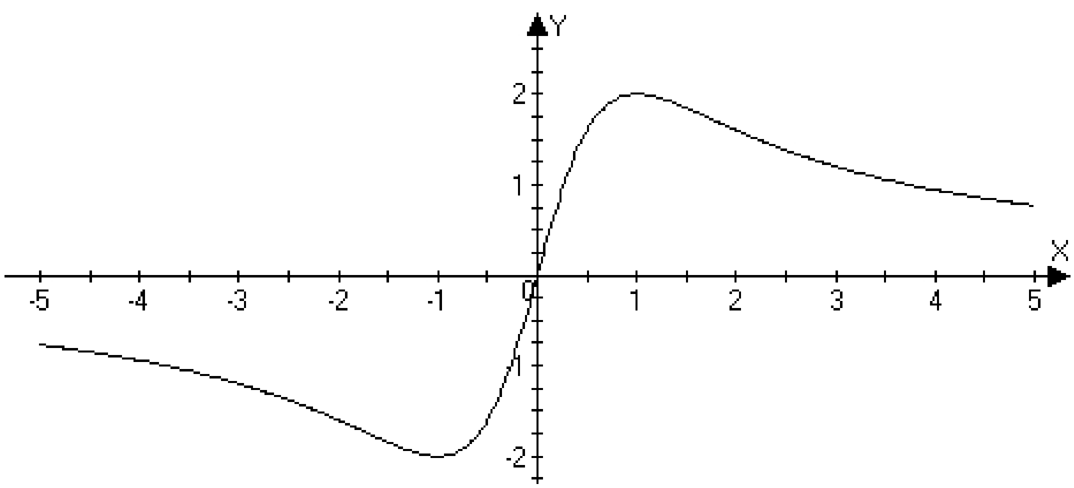


5. For what values of *a* and *b* does the function $f(x)=\dfrac{ax}{b+x^2}$ have a maximum at $f(1)=2$?

   SOLN: $f(1)=\dfrac{a}{b+1}=2 \Leftrightarrow a-2b=2$ is one linear relation.

   $$f'(1)=\left.\frac{a(b+x^2)-2ax^2}{(b+x^2)^2}\right|_{x=1}=\left.\frac{ab-ax^2}{(b+x^2)^2}\right|_{x=1}=\frac{ab-a}{(b+1)^2}=0 \Leftrightarrow b=1 \text{ and so } a=4.$$

   We know this is a maximum because the derivative $y'=\dfrac{4-4x^2}{(1+x^2)^2}$ is positive when $-1<x<1$ and negative when $x>1$. As a bonus, here's the graph of $f(x)=\dfrac{4x}{1+x^2}$:



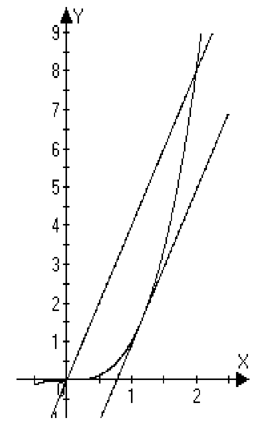
6. Find the limit:

a. $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}=\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}=\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}=\frac{n^2-m^2}{2}$

b. $\lim_{x\to -\infty} x^3e^{2x}=\lim_{x\to -\infty}\frac{x^3}{e^{-2x}}=\lim_{x\to -\infty}\frac{3x^2}{-2e^{-2x}}=\lim_{x\to -\infty}\frac{6x}{4e^{-2x}}=\lim_{x\to -\infty}\frac{6}{-8e^{-2x}}=\frac{6}{-\infty}=0$

7. Find all values of x that satisfy the conclusion of the mean value theorem for $f(x)=x^3$ on [0,2].

SOLN: $f'(c)=3c^2=\frac{f(2)-f(0)}{2-0}=\frac{8}{2}\Rightarrow c=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$



8. Show that the rectangle of largest area that can be inscribed in a circle is a square.
SOLN: Let the circle or radius r be centered on the origin in the coordinate system and let $(r\cos t, r\sin t)$ be a corner of the rectangle with area

$A(t)=(2r\cos t)(2r\sin t)=2r^2\sin 2t$ so that $A'(t)=4r^2\cos 2t=0\Leftarrow t=\frac{\pi}{4}$ so

the dimensions are $x=2r\cos\frac{\pi}{4}=\sqrt{2}r=2r\sin\frac{\pi}{4}=y$ and it's a square. So there.

9. Find the area of the largest rectangle that can fit above the x-axis and below the curve $y=e^{-x^2}$.
SOLN: Since the function is positive valued with y-axis symmetry, The area of such a rectangle will be $A(x)=(2x)\times\left(e^{-x^2}\right)=2xe^{-x^2}$. Now $A'(x)=2e^{-x^2}-4x^2e^{-x^2}=2e^{-x^2}\left(1-2x^2\right)=0$ when $x=\frac{\sqrt{2}}{2}$

whence the maximum area is $A\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}e^{-1/2}=\sqrt{\frac{2}{e}}$

10. Find an inflection point for $f(x)=\ln(2+\sin 3x)$

SOLN: $f'(x)=\frac{3\cos 3x}{2+\sin 3x}\Rightarrow f''(x)=\frac{-9\sin 3x(2+\sin 3x)-9\cos^2 3x}{(2+\sin 3x)^2}=\frac{-18\sin 3x-9}{(2+\sin 3x)^2}$ changes sign

where $\sin 3x=-\frac{1}{2}\Leftarrow 3x=-\frac{\pi}{6}\Leftrightarrow \boxed{x=-\frac{\pi}{18}}$