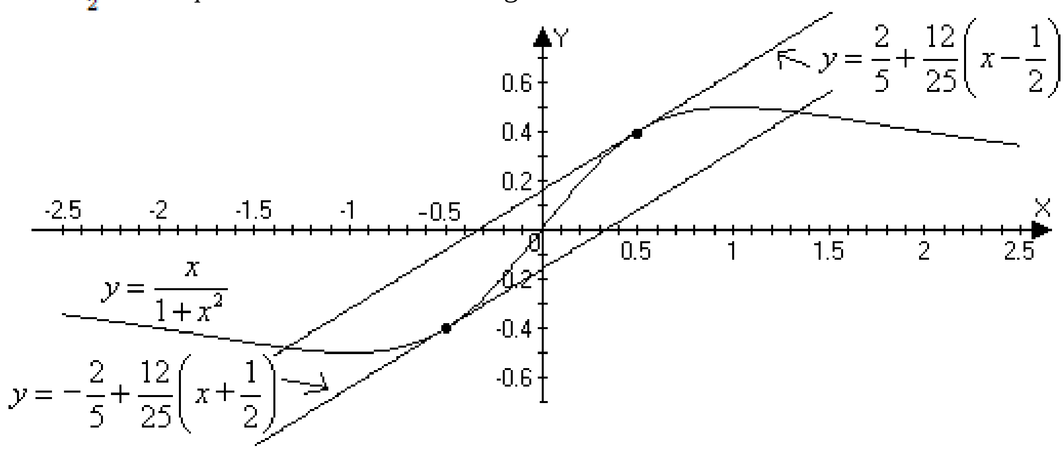
Math 1A – Chapter 3 Test – Typical Problems Set Solutions

1. Use the definition of the derivative to compute each of the following limits.
   a. $\lim_{x\to 0}\frac{(x^6+x+1)-1}{x}=\lim_{x\to 2}\frac{f(x)-f(0)}{x-0}=f'(0)=\left(6x^5+1\right)\big|_{x=0}=1$
   b. $\lim_{x\to 1}\frac{2^{3x}-8}{x-1}=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=f'(1)=(\ln 8)8^x\,|_{x=1}=8\ln 8$
   c. $\lim_{x\to 1/3}\frac{2\cos(\pi x)-1}{x-\frac{1}{3}}=\lim_{x\to 1/3}\frac{f(x)-f\left(\frac{1}{3}\right)}{x-\frac{1}{3}}=f'\left(\frac{1}{3}\right)=-2\pi\sin(\pi x)\big|_{x=\frac{1}{3}}=-\pi\sqrt{3}$
2. Find equations of the lines tangent to the curve $y=\frac{x}{1+x^2}$ which are parallel to the line $y-0.48x=0$. Sketch a graph illustrating these tangencies.
   SOLN:
   $y'=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}=0.48=\frac{12}{25}\Leftrightarrow 12(1+x^2)^2-25+25x^2=0\Leftrightarrow 12x^4+49x^2-13=0$

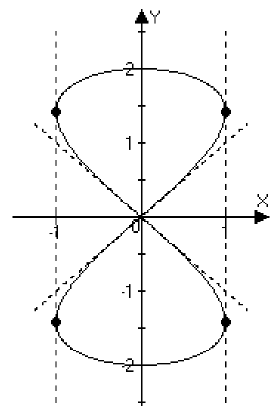
   The discriminant of this quadratic is $49^2-4(12)(-13)=2401+624=3025=55^2$ so the positive solutions are $x^2=\frac{-49+55}{24}=\frac{1}{4}$ whence the slope of a tangent line is $0.48=\frac{12}{25}$ when $x=\pm\frac{1}{2}$. The equations label the these tangents below:



3. Newton's law of gravitation says that the magnitude *F* of the force exerted by a body of mass *m* on a body of mass *M* is
   $F=\frac{GMm}{r^2}$
   a. Find *dF*/*dr* and write a sentence of two explaining what that means.
      SOLN: $\frac{dF}{dr}=-\frac{2GMm}{r^3}$ is the rate of change in the force of gravitational attraction per change in distance between the masses. Note that *dF/dr* is negative so *F* decreases as *r* increases and that the rate of change is inversely proportional to the cube of *r*.
4. Use the *definition* of the derivative to simplify $\frac{d}{dx}\cos(2x)$.
   SOLN:
   $\frac{d}{dx}\cos(2x)=\lim_{h\to 0}\frac{\cos(2x+2h)-\cos(2x)}{h}=\lim_{h\to 0}\frac{\cos(2x)\cos(2h)-\sin(2x)\sin(2h)-\cos(2x)}{h}=$
   $\lim_{h\to 0}\frac{\cos(2x)(\cos(2h)-1)-\sin(2x)\sin(2h)}{h}=\cos(2x)\lim_{h\to 0}\frac{\cos(2h)-1}{h}-2\sin(2x)\lim_{2h\to 0}\frac{\sin(2h)}{2h}$

$= 0 - 2\sin(2x)$

5. For what values of x does the graph of $f(x) = x^3 + 6x^2 + x + 4$ have a horizontal tangent?
   SOLN: $f'(x) = 3x^2 + 12x + 1 = 3(x+2)^2 - 11 = 0 \Leftrightarrow x = -2 \pm \frac{\sqrt{33}}{3}$
6. A curve C is defined by the parametric equations $= \sin(2t)$; $y = 2\cos(t)$.
   a. Show that C has two tangent lines at the origin and find their equations.
      SOLN: The curve passes through (0,0) when t is any odd multiple of π over 2: $t = \frac{(2k+1)\pi}{2}$
      The slopes of the tangent lines at $t = \frac{\pm\pi}{2}$ are $\frac{dy}{dx}\Big|_{t=\pm\frac{\pi}{2}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\Big|_{t=\pm\frac{\pi}{2}} = \frac{-2\sin(t)}{2\cos(2t)}\Big|_{t=\pm\frac{\pi}{2}} = \pm 1$.
      Thus the tangent lines are $y = \pm x$.
   b. Find the points where the tangent line in the x-y plane is vertical.
      SOLN: The tangent line will be vertical when
      $\frac{dx}{dt} = 0 \Longleftrightarrow 2\cos(2t) = 0 \Longleftrightarrow t = \frac{(2k+1)\pi}{4}$. In particular,
      $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ correspond to the points
      $(x,y) = (1,\sqrt{2}),(-1,-\sqrt{2}),(1,-\sqrt{2}),(-1,\sqrt{2})$.



7. Find an equation for the line tangent to $x^2y + xy^2 = 2xy$ at $(1,1)$.
   SOLN: Equate derivatives with respect to x of the left and right sides of the equation: $2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$ and plug in the coordinates for the point of interest:
   $2 + \frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 2 + 2\frac{dy}{dx}$ and solve for $\frac{dy}{dx} = -1$.
   Thus the tangent line is $y = 2 - x$. Note that the equation can be simplified by dividing through by $y \neq 0$ and $x \neq 0$: $x + y = 2$.
   That's much simpler, huh?

8. Let $y = \sqrt[3]{x}$
   a. Find the differential dy.
      SOLN: $dy = \frac{dx}{3\sqrt[3]{x^2}}$
   b. Evaluate dy and Δy if $x = 8$ and $= \Delta x = 0.1$
      SOLN: $dy = \frac{dx}{3\sqrt[3]{x^2}} = \frac{1}{3}\left(\frac{1}{4}\right)\left(\frac{1}{10}\right) = \frac{1}{120} = 0.08\overline{3}$
   c. Estimate $\sqrt[3]{8.1}$ using the line tangent to $= \sqrt[3]{x}\ at\ (8,2)$. What is the relative error in your estimation?
      SOLN: $\sqrt[3]{8.1} \approx 2 + 0.08\overline{3} = 2.08\overline{3}$ The relative error is an overestimation of $\frac{dy - \Delta y}{y} \approx \frac{0.08\overline{3} - 0.0082986}{2} \approx 1.724 \times 10^{-5}$
9. Find the derivative of the function $f(x) = (\cos(x))^x$ by first differentiating $\ln y$
   SOLN:
   $\frac{y'}{y} = \frac{d}{dx} x\ln\cos(x) = \ln\cos(x) - \frac{x\sin(x)}{\cos(x)} \Leftrightarrow y' =$
   $(\cos(x))^x \ln\cos(x) - x\sin(x)\,(\cos(x))^{x-1}$

10. Consider $f(r) = 2\sqrt{r} - 3\sqrt[3]{r}$

    a. Simplify formulas for the first and second derivatives of $f(r)$

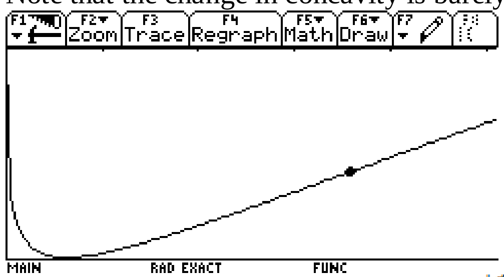
    SOLN: $f'(r) = \frac{1}{\sqrt{r}} - \frac{1}{\sqrt[3]{r^2}}$ and $''(r) = \frac{-1}{2\sqrt{r^3}} + \frac{2}{3\sqrt[3]{r^5}}$.

    b. Find the inflection point for $f(r)$

    SOLN: The inflection point is where the second derivative changes sign.

    $-\frac{1}{2\sqrt{r^3}} + \frac{2}{3\sqrt[3]{r^5}} = 0 \Leftrightarrow \frac{r^{-3/2}}{2} = \frac{2r^{-5/2}}{3} \Leftrightarrow r^{1/6} = \frac{4}{3} \Leftrightarrow r = \left(\frac{4}{3}\right)^6 \approx 5.62$

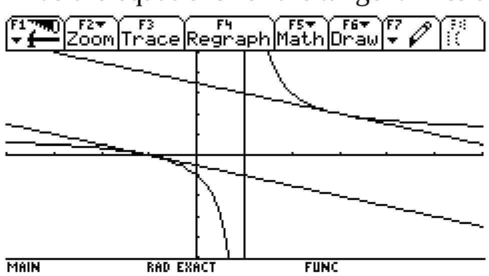
    Note that the change in concavity is barely discernable in the graph:



11. Find equations for the tangent lines to $y = \frac{x+1}{x-1}$ that are parallel to the line $+2y = 17$.

    SOLN: $y = \frac{x+1}{x-1} = 1 + \frac{2}{x-1} \Rightarrow y' = -\frac{2}{(x-1)^2}$, so the slope of the tangent line is parallel to the given line where $y' = -\frac{1}{2} \Leftrightarrow -\frac{2}{(x-1)^2} = -\frac{1}{2} \Leftrightarrow (x-1)^2 = 4 \Leftrightarrow x = 1 \pm 2 \Leftrightarrow y = 1 \pm 1$

    Thus the equations for the tangent lines are $y = 2 - \frac{1}{2}(x-3)$ and $y = -\frac{1}{2}(x+1)$.



12. If $h(9) = 3$ and $h'(9) = -7$, find $\frac{d}{dx}\left(\frac{h(x)}{x}\right)|_{x=9}$

    SOLN: $\frac{d}{dx}\left(\frac{h(x)}{x}\right)|_{x=9} = \left(\frac{xh'(x) - h(x)}{x^2}\right)|_{x=9} = \frac{9(-7)-3}{9^2} = -\frac{22}{27}$

13. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

    $$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_{max}}\right)P(t) - H \cdot P(t)$$

    where r_0 is the birth rate of the fish, P_{max} is the maximum population the pond can sustain and H is the proportion of fish harvested in a year. If the pond can sustain a maximum population of 5000 fish, the birth rate is 4% and the harvesting rate is 2%, what (nonzero) population level(s) weill not change, according to the model.

    SOLN: The population will not change when $\frac{dP}{dt} = 0$. Plugging in the parameter values,

    $0.04\left(1 - \frac{P}{5000}\right)P - 0.02P = 0 \Leftrightarrow 0.02P - 0.000008P^2 = 0$, so either $P = 0$ or

    $P = \frac{0.02}{0.000008} = 2500$.

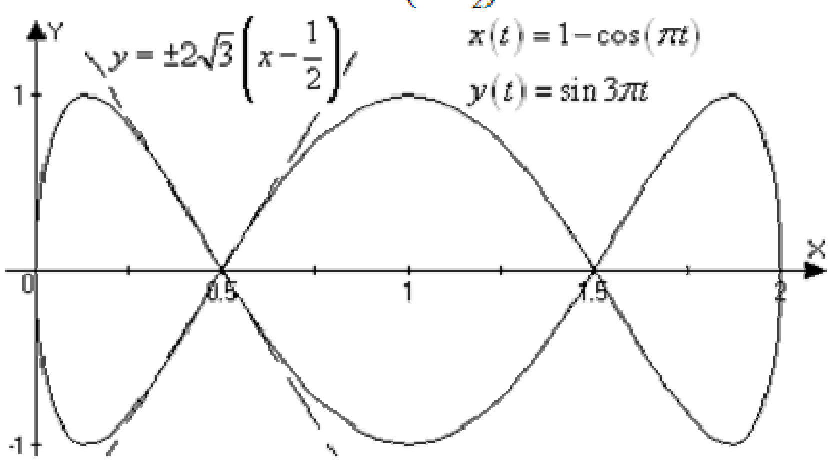
14. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8$ atm and is increasing at a rate of 0.14 atm/min and $V = 11$L and is decreasing at a rate of 0.17 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ moles. Round your answer to four decimal places.
SOLN: Equating derivatives of left and right sides of the gas law with respect to t,

$$nR\frac{dT}{dt} = \frac{d}{dt}PV = P\frac{d}{dt}V + V\frac{d}{dt}P \Leftrightarrow 0.821\frac{d}{dt}T = 8(-17) + 11(0.14) = -136 + 1.54$$
$$\Leftrightarrow \frac{d}{dt}T = -163.8$$

15. Use the definition of the derivative to compute $\frac{d}{dx}\sec(x)$.
SOLN:

$$\frac{d}{dx}\sec(x) = \lim_{h\to 0}\frac{\sec(x+h)-\sec(x)}{h} =$$
$$\lim_{h\to 0}\frac{\frac{1}{\cos(x)\cos(h)-\sin(x)\sin(h)}-\frac{1}{\cos(x)}}{h}\frac{(\cos(x)\cos(h)-\sin(x)\sin(h))\cos(x)}{(\cos(x)\cos(h)-\sin(x)\sin(h))\cos(x)} =$$
$$\lim_{h\to 0}\frac{\cos(x)-(\cos(x)\cos(h)-\sin(x)\sin(h))}{h(\cos(x)\cos(h)-\sin(x)\sin(h))\cos(x)} = \lim_{h\to 0}\frac{\cos(x)(1-\cos(h))+\sin(x)\sin(h)}{h(\cos(x)\cos(h)-\sin(x)\sin(h))\cos(x)} =$$
$$\cos(x)\lim_{h\to 0}\frac{1-\cos(h)}{h} + \sin(x)\lim_{h\to 0}\frac{\sin(h)}{h(\cos(x)\cos(h)-\sin(x)\sin(h))\cos(x)} = 0 + \frac{\sin(x)}{\cos^2(x)} =$$
$$\tan(x)\sec(x)$$

16. Show that the curve described by the parametric equations
$x(t) = 1 - \cos(\pi t);\ \ y(t) = \sin(3\pi t)$
has two tangent lines where $t = 1/2$ and find their equations. Illustrate these in a graph.
SOLN: $x = ½$ means that $1 - \cos(\pi t) = \frac{1}{2} \Leftrightarrow \cos(\pi t) = \frac{1}{2} \Leftrightarrow \pi t = \pm\frac{\pi}{3} = 2\pi k$ where $k \in \mathbb{Z}$.
With $k = 0$, $t = \pm\frac{1}{3}$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\pi\cos(3\pi t)}{\pi\sin(\pi t)}$. At $= \frac{1}{3}$, $\frac{dy}{dx} = \frac{-3\pi}{\pi\sqrt{3}/2} = \frac{-6}{\sqrt{3}} = -2\sqrt{3}$ so the tangent line has equation $y = -2\sqrt{3}\left(x - \frac{1}{2}\right)$. At $t = -\frac{1}{3}$, $\frac{dy}{dx} = \frac{-3\pi}{-\pi\sqrt{3}/2} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$ so the tangent line has equation $y = 2\sqrt{3}\left(x - \frac{1}{2}\right)$



17. If $\sqrt{3}\sin(x)\cos(y) = 1$, find a formula for $\frac{dy}{dx}$ using implicit differentiation.

SOLN: $\frac{d}{dx}\sqrt{3}\sin(x)\cos(y) = 0 \Leftrightarrow \sqrt{3}\cos(x)\cos(y) - \sqrt{3}\sin(x)\sin(y)\frac{dy}{dx} = 0 \Leftrightarrow$
$\frac{dy}{dx} = \frac{1}{\tan(x)\tan(y)}$

18. Find $\frac{d^{63}}{dx^{63}}(x\sin(x))$ by finding the first few derivatives and deducing the pattern that occurs.

SOLN:

n	1	2	3	4
$\frac{d^n}{dx^n}(x\sin x)$	$\sin x + x\cos x$	$2\cos x - x\sin x$	$-3\sin x - x\cos x$	$-4\cos x + x\sin x$
n	5	6	7	8
$\frac{d^n}{dx^n}(x\sin(x))$	$5\sin x + x\cos x$	$6\cos x - x\sin x$	$-7\sin x - x\cos x$	$-8\cos x + x\sin x$

Evidently there's a 4-cycle. Since $63 = 4 \cdot 15 + 3$ we look to the third entry in the pattern and deduce that $\frac{d^{63}}{dx^{63}}(x\sin(x)) = -63\sin x - x\cos x$

19. Use logarithmic differentiation to find the derivative of $y = (\sec x)^x$
SOLN:
$\frac{d}{dx}\ln y = \frac{y'}{y} = \frac{d}{dx}(x\ln\sec x) = \ln\sec x + x\left(\frac{\sec x\tan x}{\sec x}\right) = \ln\sec x + x\left(\frac{\sec x\tan x}{\sec x}\right) =$
$x\tan x - \ln\cos x$
so that $y' = x(\sec x)^x\tan x - (\sec x)^x\ln\cos x$

20. Verify the given linearization $\ln|x = 1| \approx x$ at $a = 0$. Then determine the values of $x$for which the linear approximation is accurate to within 0.1
SOLN: Near $x = 0$, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1 \cdot x = x$. To find where the error in approximation is no more than 0.1, solve $|\ln|x+1| - x| \le 0.1 \Leftarrow -0.3832 \le x \le 0.5162$
Of course you'd need a calculator (say the TI85) to find the roots of

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y1=abs(x-ln(abs(x+1)))-0.1:
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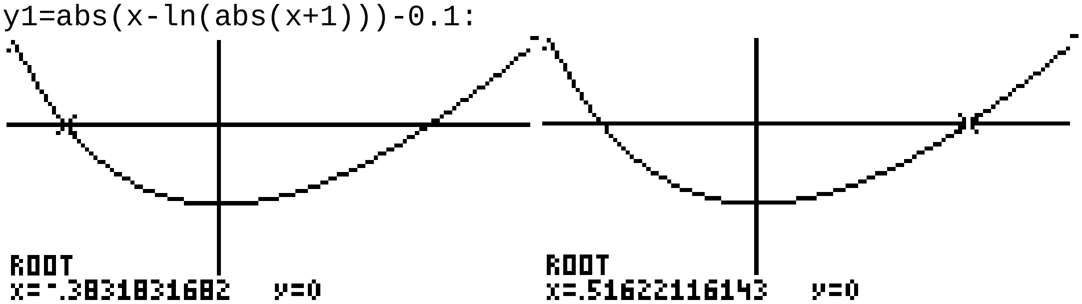
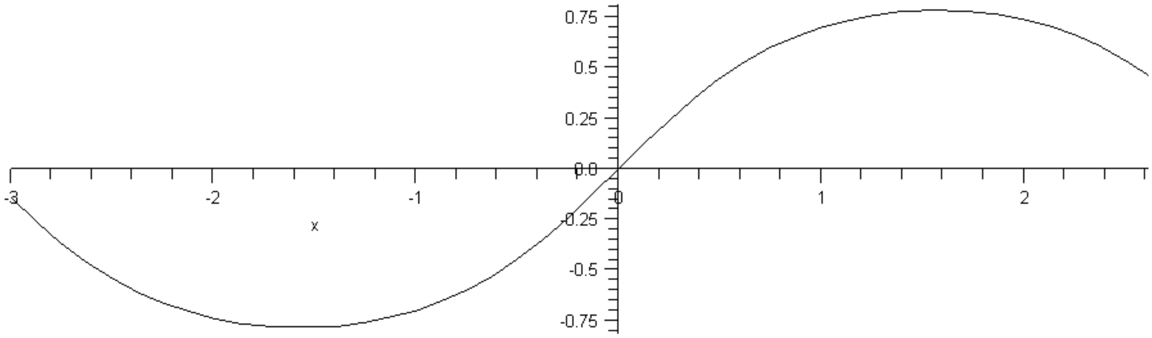


21. Find an equation for the line tangent to $y = \tan^{-1}(\sin x)$ where $x = 0$.
SOLN: The point of tangency is at the origin since $y = \tan^{-1}(\sin 0) = \tan^{-1} 0 = 0$. The slope is $\frac{dy}{dx}|_{x=0} = \frac{d}{dx}\sin x \frac{d}{du}\tan^{-1}u = \cos x \frac{1}{1+\sin^2 x}|_{x=0} = 1$. So the tangent line is $y = x$.

This seems quite reasonable, given the graph of $y = \tan^{-1}(\sin x)$ here:

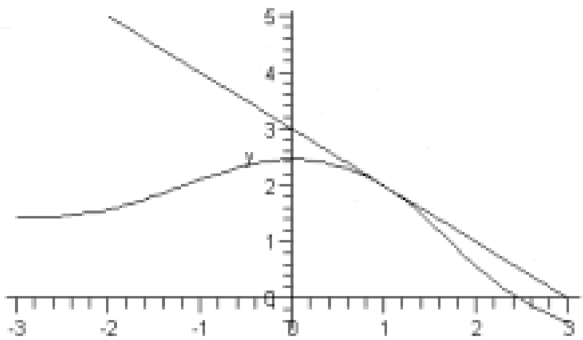


22. Find an equation of the tangent line to the parametric curve $x = e^{\sqrt{t}},\ y = -t^6 + t$

At the point corresponding to $t = 1$.
SOLN: The point of tangency has coordinates $(e,0)$ and the slope is
$\frac{dy}{dx}\Big|_{t=1} = \frac{dy/dt}{dx/dt}\Big|_{t=1} = \frac{-6t^5+1}{e^{\sqrt{t}}/2\sqrt{t}}\Big|_{t=1} = -\frac{5}{e/2} = -\frac{10}{e}$, so the equation is
$y = -\frac{10}{e}(x-e) = 10 - \frac{10x}{e}$

23. Find y'' if $x^8 + y^8 = 1$.
SOLN: Differentiating implicitly, $8x^7 + 8y^7y' = 0 \Leftarrow y' = -\frac{x^7}{y^7} \Rightarrow$
$y'' = -\frac{7y^7x^6 - 7x^7y^6y'}{y^{14}} = -\frac{7y^7x^6 + x^7y^6\left(\frac{x^7}{y^7}\right)}{y^{14}} = -\frac{7x^6(x^8+y^8)}{y^{15}} = -\frac{7x^6}{y^{15}}$

24. Find an equation for the line tangent to the curve $y^3 + 3x^2y + x^3 = 15$ at (1,2).
SOLN: $3y^2\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} + 3x^2 = 0$ and substituting (1,2) for (x,y) we have
$12\frac{dy}{dx} + 12 + 3\frac{dy}{dx} + 3 = 0 \Leftrightarrow y' = -1$ so the tangent line is $y = 2 - (x-1) = 3 - x$
As a bonus, check out the graph illustrating the line tangent to the curve:

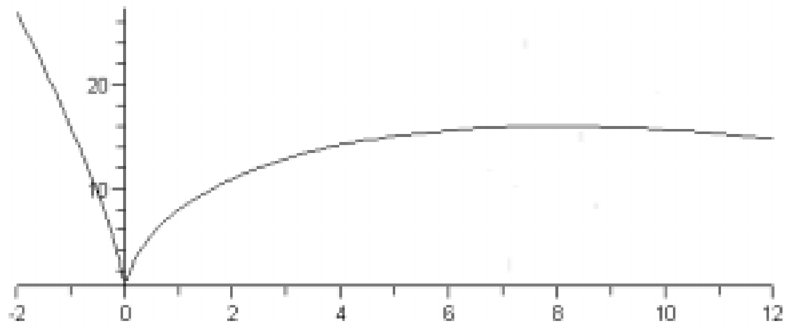


25. Use implicit differentiation to find an equation of the line tangent to $y^2\cos\left(\frac{\pi x}{8}\right) = xy - 8$ at (4,2).
SOLN: Equating derivatives of left and right sides,

$2y\dfrac{dy}{dx}\cos\left(\dfrac{\pi x}{8}\right)-\dfrac{\pi y^2}{8}\sin\left(\dfrac{\pi x}{8}\right)=y+x\dfrac{dy}{dx}$ and plugging in the given coordinates we have $0-\dfrac{\pi}{2}=2+4\dfrac{dy}{dx}\Leftrightarrow\dfrac{dy}{dx}=-\dfrac{4+\pi}{8}$ so the tangent line is $y=2-\dfrac{4+\pi}{8}(x-4)$

26. At what point on the curve $y=12x^{2/3}-4x$ is the tangent line horizontal?

SOLN: $\dfrac{dy}{dx}=8x^{-1/3}-4=0\Leftrightarrow x=8$ where $y = 16$. Here's a graph:



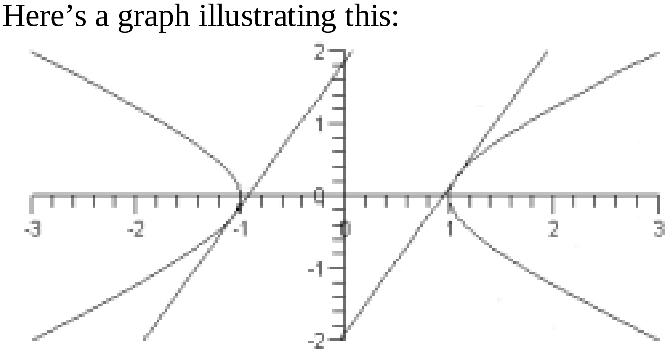
27. Derive a simple formula for the derivative $\dfrac{d}{dx}\sinh^{-1}x$

SOLN: Note first that $y=\sinh^{-1}x\Leftrightarrow x=\sinh y$. Then consider the basic identity for the hyperbolic sine $\cosh^2 y=1+\sinh^2 y=1+x^2$ so $\dfrac{dy}{dx}=\dfrac{1}{dx/dy}=\dfrac{1}{\cosh y}=\dfrac{1}{\sqrt{1+x^2}}$.

28. Find the points on the hyperbola $x^2-2y^2=1$ where the tangent line has slope = 2.

SOLN: Differentiating implicitly, $2x-4y\dfrac{dy}{dx}=0$. Substituting 2 for the slope and solving for y we have $x=4y$. Substituting this into the equation, we have $16y^2-2y^2=1\Leftrightarrow y=\pm\dfrac{\sqrt{14}}{14}\Leftrightarrow x=\dfrac{2\sqrt{14}}{7}$ i.e. at $\left(-\dfrac{2\sqrt{14}}{7},\dfrac{\sqrt{14}}{14}\right),\left(\dfrac{2\sqrt{14}}{7},-\dfrac{\sqrt{14}}{14}\right)$.

Here's a graph illustrating this:



29. Let $\begin{array}{l}x=\sin 4t-\cos t\\ y=\cos 3t-\sin t\end{array}$.

a. Find $\dfrac{dy}{dx}$ as a function of t.

SOLN: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-3\sin 3t - \cos t}{4\cos 4t + \sin t}$

b. Find $\dfrac{d^2y}{dx^2}$ as a function of t.

SOLN:

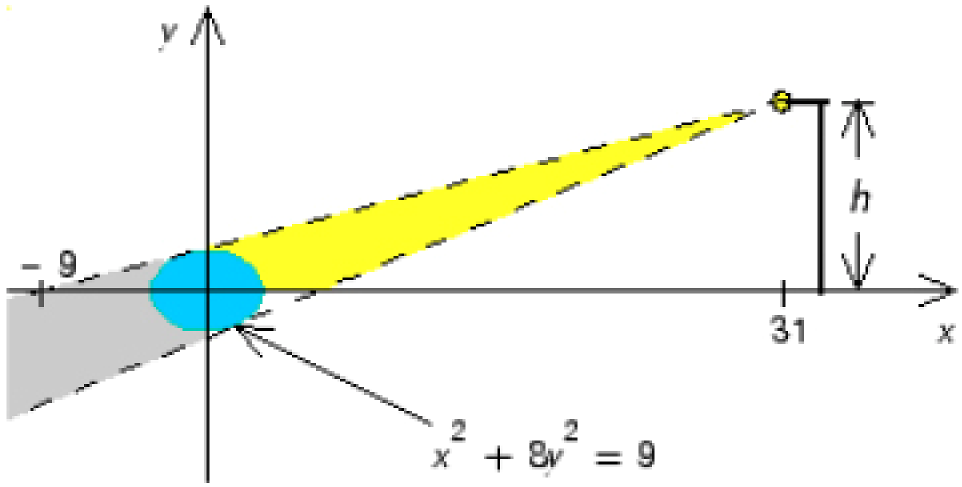
$$\frac{d}{dx}\frac{dy}{dx} = \frac{dt}{dx}\frac{d}{dt}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\frac{-3\sin 3t - \cos t}{4\cos 4t + \sin t}}{4\cos 4t + \sin t}$$

$$= \frac{(4\cos 4t + \sin t)(-9\cos 3t + \sin t) - (-16\sin 4t + \cos t)(-3\sin 3t - \cos t)}{(4\cos 4t + \sin t)^3}$$

$$= \frac{1 - 36\cos 4t\cos 3t + 4\cos 4t\sin t - 9\cos 3t\sin t - 48\sin 4t\sin 3t - 16\sin 4t\cos t +}{(4\cos 4t + \sin t)^3}$$

30. Find a parabola that passes through (1,10) and whose tangent lines at $x = -2$ and $x = 1$ have
slopes –5 and 7, respectively.
SOLN: Let $y = ax^2 + bx + c$ so that $y' = 2ax + b$. First substitute into this second form to get the 2X2 system $\begin{aligned} -4a + b &= -5 \\ 2a + b &= 7 \end{aligned}$ whose solution is $a = 2$ and $b = 3$. Now use the information about what point lies on the parabola to solve for c:
$10 = a + b + c = 2 + 3 + c \Rightarrow c = 5$. Thus $y = 2x^2 + 3x + 5$ is the parabola we seek.

31. The figure shows a lamp located 31 units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 8y^2 \le 9$. If the point $(-9, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



SOLN: The coordinates of the lamp are (31,h) and the upper shadow line has slope $m = \frac{h}{40}$ so that the line tangent to the top of the ellipse at $\left(a, \frac{1}{4}\sqrt{18-2a^2}\right)$. We can find the slope in terms of a by differentiating implicitly and also using the rise over run formula:

$$2a + 16\left(\frac{1}{4}\sqrt{18-2a^2}\right)m = 0 \Leftrightarrow m = \frac{-a}{2\sqrt{18-2a^2}} = \frac{\sqrt{18-2a^2}}{4(a+9)}$$

This gives an equation we can solve for a: $-4a^2 - 18a = 18 - 2a^2 \Leftrightarrow a = -1$

So that the slope is $\frac{h}{40} = \frac{1}{8} \Leftrightarrow h = 5$

32. Find an equation for the line tangent to the curve described by the parametric equations $\begin{array}{l} x = \sin(3t) - \cos t \\ y = \cos(3t) - \sin t \end{array}$ where $t = \frac{\pi}{6}$.

SOLN:

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \left.\frac{-3\sin(3t)-\cos t}{3\cos(3t)+\sin t}\right|_{t=\frac{\pi}{6}}$$

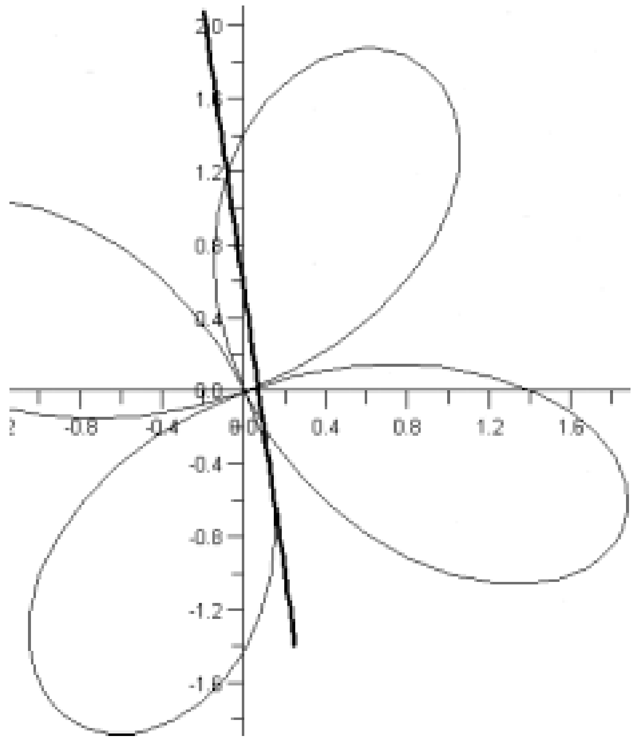
$$= \frac{-3\sin\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{6}\right)}{3\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{6}\right)}$$

$$= \frac{-3-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -6-\sqrt{3}$$

Also, at $t=\frac{\pi}{6}$, $(x, y) = \left(1-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

so that $y+\frac{1}{2} = -\left(6+\sqrt{3}\right)\left(x-1+\frac{\sqrt{3}}{2}\right)$

is an equation for the tangent line we seek.



33. A particle moves on a horizontal line so that its coordinate at time t is $x = e^{-t/4}\cos(2t)$.

a. Find the velocity and acceleration functions.

$$v = \frac{dx}{dt} = -\frac{1}{4}e^{-t/4}\cos(2t) - 2e^{-t/4}\sin(2t) = -\frac{e^{-t/4}}{4}\left(\cos(2t)+8\sin(2t)\right)$$

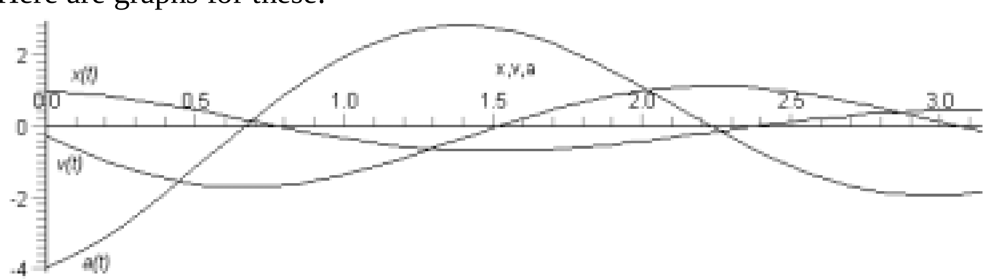
$$= -\frac{\sqrt{65}e^{-t/4}}{4}\sin\left(2t+\arctan\frac{1}{8}\right)$$

$$a = \frac{dv}{dt} = \frac{e^{-t/4}}{16}\left(\cos(2t)+8\sin(2t)\right) + \frac{e^{-t/4}}{4}\left(2\sin(2t)-16\cos(2t)\right)$$

$$= \frac{e^{-t/4}}{16}\left(16\sin(2t)-63\cos(2t)\right)$$

$$= \frac{65e^{-t/4}}{16}\sin\left(2t-\arctan\frac{63}{16}\right)$$

Here are graphs for these:



The initial velocity is negative and decreasing, so the initial acceleration is negative.

b. Find the distance the particle travels in the time $0 \le t \le \pi$.

SOLN: From the graphs of x, v, and a shown above it is clear that the position x

starts out positive and moving to the left and then somewhere near $t = 1.5$ has achieved a negative position but stopped moving left and is starting to move right. It continues moving right until near where $t = 3$ where is starts moving left again for a bit. To determine the places where the particle turns around, we solve $v = 0$:

$$-\frac{\sqrt{65}e^{-t/4}}{4}\sin\left(2t+\arctan\frac{1}{8}\right)=0 \Leftrightarrow 2t+\arctan\frac{1}{8}= \text{ some multiple of } \pi$$

$$2t+\arctan\frac{1}{8}=\pi \ or\ 2\pi \Rightarrow$$

That is,

$$t=\frac{\pi}{2}-\frac{1}{2}\arctan\frac{1}{8} \text{ or } \pi-\frac{1}{2}\arctan\frac{1}{8}$$

So the extreme left/right positions of x are at $x(0)=1$,

$$x\left(\frac{\pi}{2}-\frac{1}{2}\arctan\frac{1}{8}\right)=\exp\left(\frac{1}{8}\arctan\frac{1}{8}-\frac{\pi}{8}\right)\cos\left(\pi-\arctan\frac{1}{8}\right)$$

$$=-\frac{8}{\sqrt{65}}\exp\left(\frac{1}{8}\arctan\frac{1}{8}-\frac{\pi}{8}\right)\approx -0.6805$$

$$x\left(\pi-\frac{1}{2}\arctan\frac{1}{8}\right)=\exp\left(\frac{1}{8}\arctan\frac{1}{8}-\frac{\pi}{4}\right)\cos\left(2\pi-\arctan\frac{1}{8}\right)$$

$$=\frac{8}{\sqrt{65}}\exp\left(\frac{1}{8}\arctan\frac{1}{8}-\frac{\pi}{4}\right)\approx 0.4595$$

Finally, $x(\pi)=e^{-\pi/4}\approx 0.4559$ so the object travels about

(1+0.6805)+(0.4595+0.6805)+(0.4595–0.4559) = 2.8241

c. When is the particle speeding up? When is it slowing down?
SOLN: The particle is speeding up when the acceleration and the velocity are in the same direction. This is true initially, and continues to be true until

$$a=\frac{65e^{-t/4}}{16}\sin\left(2t-\arctan\frac{63}{16}\right)=0 \Rightarrow 2t-\arctan\frac{63}{16}=0 \Leftrightarrow t=\frac{1}{2}\arctan\frac{63}{16}\approx 0.6610$$

Then again when

$$v=-\frac{\sqrt{65}e^{-t/4}}{4}\sin\left(2t+\arctan\frac{1}{8}\right)=0 \Leftarrow 2t+\arctan\frac{1}{8}=\pi \Leftrightarrow t=\frac{\pi}{2}-\frac{1}{2}\arctan\frac{1}{8}\approx 1.509$$

it starts speeding up in the positive direction until the acceleration becomes negative again at $t=\frac{\pi}{2}+\frac{1}{2}\arctan\frac{63}{16}\approx 2.2318$ - then there's a little interval at the end when a and v are both negative after $t=\pi-\frac{1}{2}\arctan\frac{1}{8}\approx 3.079$

Summing up, the particle is speeding up on this union of intervals:

$$\left(0,\frac{1}{2}\arctan\frac{63}{16}\right)\cup\left(\frac{\pi}{2}-\frac{1}{2}\arctan\frac{1}{8},\frac{\pi}{2}+\frac{1}{2}\arctan\frac{63}{16}\right)\cup\left(\pi-\frac{1}{2}\arctan\frac{1}{8},\pi\right)$$

34. Consider $f(x)=\sqrt[3]{999+x^3}$

d. Find the linearization of at $x = 9$ and use it to approximate $\sqrt[3]{1999}$.

SOLN:

$$f(x) \approx f(9) + f'(9)(x-9) = \sqrt[3]{999+9^3} + \frac{9^2}{\left(999+9^3\right)^{2/3}}(x-9) = 12 + \frac{9}{16}(x-9)$$

When $x = 10$, $f(10) = \sqrt[3]{1999} \approx 12 + \frac{9}{16} = \frac{201}{16} = 12.5625$

e. What is the % error in your approximation?

SOLN: $\frac{\Delta y}{y} = \frac{f(10) - f(9)}{f(9)} \approx 0.0498 \approx \frac{dy}{y} = \frac{f'(9)dx}{f(9)} = \frac{9/16}{12} = \frac{3}{64} = 0.046875$

35. Find the coordinates of the points where a line through (0,3) is tangent to the unit circle.

SOLN: The point of tangency is on the upper half of the circle so its coordinates have the form (x, y)

So the slope of a line connecting this point with (0,3) must be $\frac{y-3}{x-0}$. The slope of a line tangent to the unit circle can also be found by implicit differentiation:

$2x + 2yy' = 0 \Leftrightarrow y' = -\frac{x}{y}$ so $\frac{y-3}{x} = -\frac{x}{y} \Leftrightarrow x^2 + y^2 = 3y$. Since the point is on the unit circle, this means that $1 = 3y \Leftrightarrow y = \frac{1}{3} \Leftrightarrow x^2 = \frac{8}{9} \Leftrightarrow x = \pm\frac{2\sqrt{2}}{3}$.

The points are thus $\left(\pm\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$.