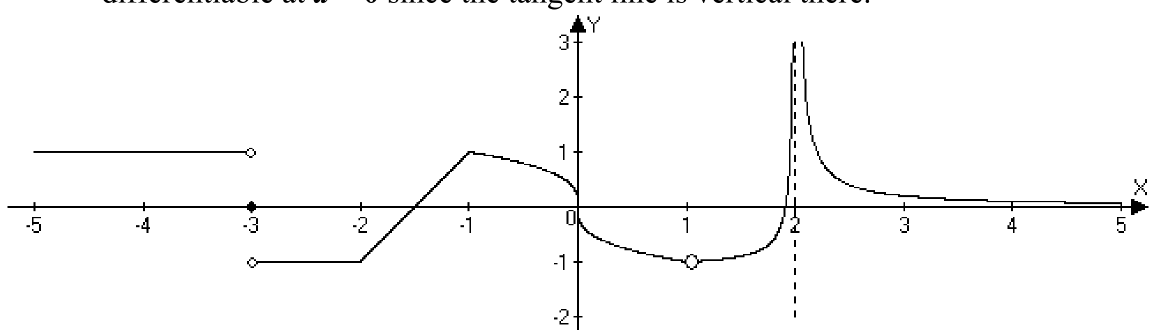
Math 1A – Final Exam Solutions – Fall '04

1. The graph below shows $y = f(x)$.
   a. State, with reasons, the numbers a at which $\lim_{x\to a} f(x)$ does not exist.

      SOLN: The limit does not exist at $\boldsymbol{a} = \mathbf{-3}$ since the limit from the left is 1 while the limit from the right is -1. Also, the limit does not exist at $\boldsymbol{a} = \mathbf{2}$ since the value of f is unbounded above both from the left and the right.
   b. State where the function is discontinuous and classify the discontinuity as removable, jump or asymptote.
      SOLN: There is a jump discontinuity at $\boldsymbol{a} = \mathbf{-3}$, a removable discontinuity at $\boldsymbol{a} = \mathbf{1}$ and a vertical asymptote discontinuity at $\boldsymbol{a} = \mathbf{2}$.
   c. State, with reasons, the numbers at which the function is not differentiable.
      SOLN: The function is not differentiable where $\boldsymbol{a} = \mathbf{-3, 1}$ and **2**, since these are *discontinuities*. The function is not differentiable at $\boldsymbol{a} = \mathbf{-2}$ since the slope is 0 as approached from the left and 2 as approached from the right. The function is no differentiable at $\boldsymbol{a} = \mathbf{-1}$ since the slope is a constant 2 as approached from the left while the slope approaches some negative value from the right. The function is not differentiable at $\boldsymbol{a} = \mathbf{0}$ since the tangent line is vertical there.

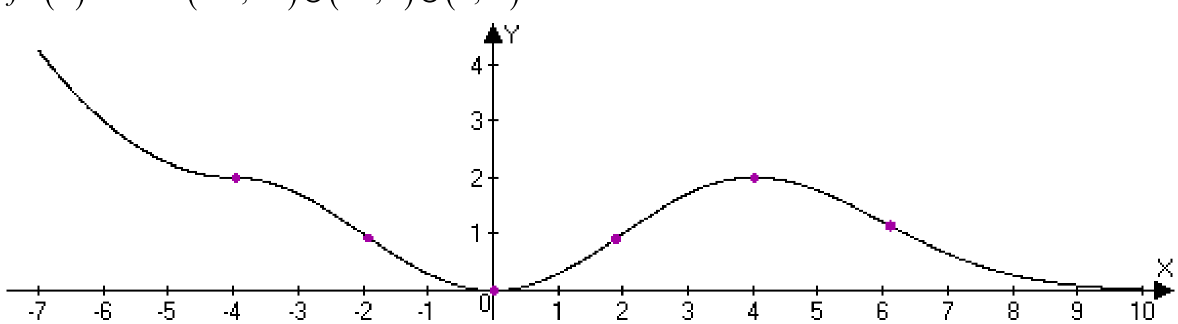


2. Suppose $f(x)$ is a function such that for all $x > 1$, $\frac{2}{\pi}\arctan x < f(x) < \frac{x+1}{x-1}$.

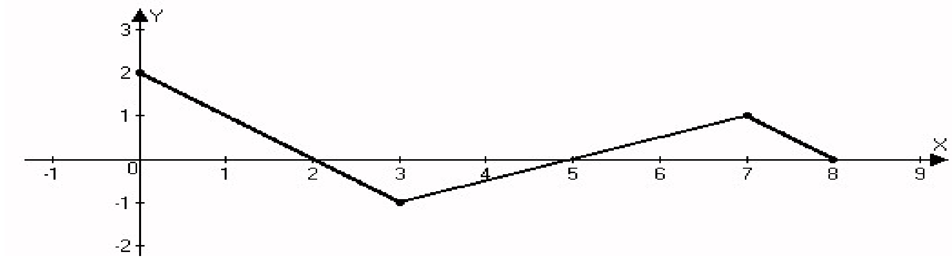
   What is $\lim_{x\to\infty} f(x)$? Why?

   SOLN: Since $\lim_{x\to\infty}\frac{2}{\pi}\arctan x = 1^-$ approaches 1 from below and $\lim_{x\to\infty}\frac{x+1}{x-1} = 1^+$ approaches 1 from above, we can conclude that a function caught between these must also converge to 1.

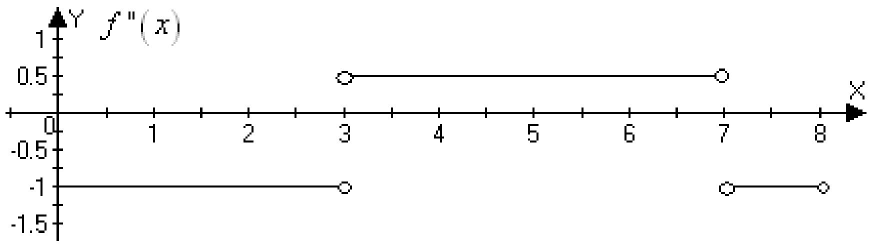
3. Sketch a graph for a function that meets the given conditions:
   $f(0) = 0, \quad f'(-4) = f'(0) = f'(4) = 0$
   $f'(x) < 0$ on $(-\infty,-4)\cup(-4,0)\cup(4,\infty)$
   $f''(x) > 0$ on $(-\infty,-4)\cup(-2,2)\cup(6,\infty)$



4. The graph of $f'(x)$ is shown below.

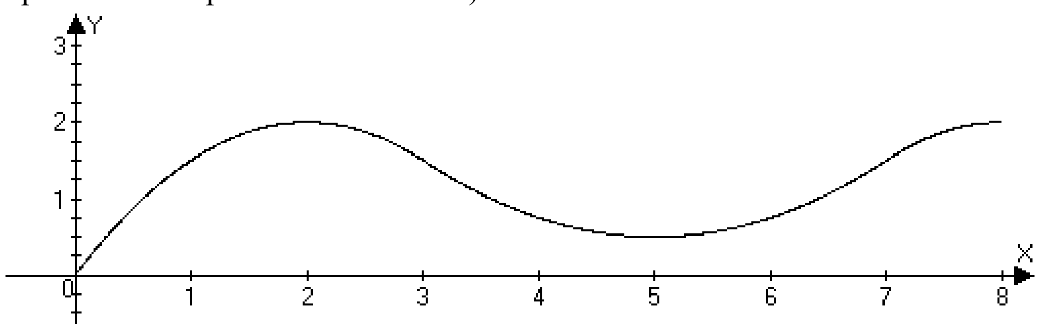


a. Sketch a graph for $f''(x)$. SOLN:



b. Sketch a possible graph for $f(x)$.

SOLN (note that only the essential ingredients of increasing/decreasing and concave up/down are important to show here):



5. Find a linear approximation for the function $f(x) = \sin\left(x + \frac{\pi}{3}\right)$ at $a = 0$ and use it to approximate $f(-0.14)$.

SOLN: In a neighborhood of $x = 0$, $f(x) = f(0) + f'(0)x = \frac{\sqrt{3}}{2} + \frac{x}{2}$ so that

$f(-0.14) = \frac{\sqrt{3}}{2} - \frac{0.14}{2} \approx 0.796$

6. Find the slope of the line tangent to the curve $x^2 \cos y + \sin 2y = xy$ at $(x, y) = \left(0, \frac{\pi}{2}\right)$.

SOLN: Assuming y is a function of x and equating derivatives,

$$\frac{d}{dx}\left(x^2 \cos y + \sin 2y\right) = \frac{d}{dx}(xy)$$

$$2x\cos y - x^2 \sin y\frac{dy}{dx} + 2\cos(2y)\frac{dy}{dx} = y + x\frac{dy}{dx}$$

Plugging in the given values: $0 - 0 + 2\cos(\pi)\frac{dy}{dx} = \frac{\pi}{2} + 0 \Leftrightarrow \frac{dy}{dx} = -\frac{\pi}{4}$

7. Find an equation for the line tangent to $y = e^{\sin 3x}$ at $x = \frac{\pi}{3}$.

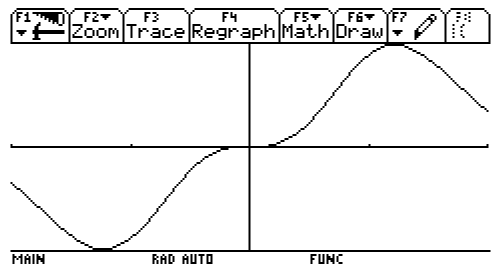
SOLN: Here $y = e^{\sin \pi} = 1$ and the slope is $y'\big|_{x=\pi/3} = 3\cos 3x e^{\sin 3x}\big|_{x=\pi/3} = -3$ so the equation could be $y - 1 = -3\left(x - \frac{\pi}{3}\right) \Leftrightarrow y = -3x + \pi + 1$

8. At what point on the curve $y = \left[\ln(x+4)\right]^2$ is the tangent line horizontal?

SOLN: $y' = \frac{2\left[\ln(x+4)\right]}{x+4} = 0 \Leftrightarrow \ln(x+4) = 0 \Leftrightarrow \boxed{x = -3}$

9. Find the local and global extreme values of the function $y = x^3 e^{-x^2}$ on the interval [-1,2].

SOLN:



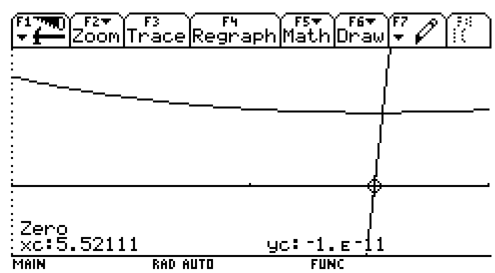
$y' = 3x^2 e^{-x^2} - 2x^4 e^{-x^2} = x^2 e^{-x^2}\left(3 - 2x^2\right) = 0 \Leftrightarrow x = \pm\sqrt{\frac{3}{2}}$

or $x = 0$. Since y'<0 only on $\left(-\infty, -\sqrt{\frac{3}{2}}\right) \cup \left(\sqrt{\frac{3}{2}}, \infty\right)$,

y has a global min at the left endpoint $x = -1$ and a local min where $x = 2$. y has a global max where $x = \sqrt{\frac{3}{2}}$. There is an inflection point where $x = 0$, but since y is increasing both before and after this point, there is no max nor min at the origin.

10. Find the point on the hyperbola $xy = 16$ that is closest to the point (4,0).

SOLN: A point on the hyperbola can be described by $\left(x, \frac{16}{x}\right)$.



The square of the distance from (4,0) to that point is

$D^2 = (x-4)^2 + \left(\frac{16}{x} - 0\right)^2 = x^2 - 8x + 16 + \frac{256}{x^2}$. Differentiating,

$2D\frac{dD}{dx} = 2x - 8 - \frac{512}{x^3} = \frac{2x^4 - 8x^3 - 512}{x^3} = 0 \Leftrightarrow x^4 - 4x^3 - 256 = 0$

Sketching both the square of the distance and its rate of change per change in x and using the calculator to find the zero of the latter, we see the distance is minimized where $x = 5.52111$, whence $y = 16/5.52111 = 2.898$.

11. Evaluate the limit: $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x}$.

SOLN: Since both the numerator and the denominator are differentiable functions approaching zero, $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x} = \lim_{x \to 0} \frac{ae^{ax} - be^{bx}}{1} = a - b$

12. The velocity of a wave of length L in deep water is $v = k\sqrt{\frac{L}{C}+\frac{C}{L}}$ where k and C are known positive constants. What is the length of the wave that gives the minimum velocity?

SOLN: $\frac{dv}{dL} = \frac{k}{2}\left(\frac{L}{C}+\frac{C}{L}\right)^{-1/2}\left(\frac{1}{C}-\frac{C}{L^2}\right) = 0 \Leftrightarrow \frac{1}{C} = \frac{C}{L^2} \Rightarrow L = C$

13. What is the maximum slope of a line connecting the origin (0,0) with a point on the parabola $y = 1-(x-2)^2$?

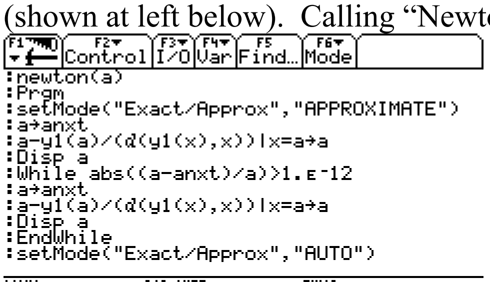
SOLN: $m(x) = \frac{1-(x-2)^2-0}{x-0} = -x+4-\frac{3}{x} \Rightarrow m'(x) = -1+\frac{3}{x^2} = 0 \Leftrightarrow x = \pm\sqrt{3}$. These correspond to the local max at $\left(\sqrt{3}, 4-2\sqrt{3}\right)$ and a local min at $\left(-\sqrt{3}, 4+2\sqrt{3}\right)$, but the slope is unbounded above, as seen by looking at $\lim_{x\to 0^-} m(x) = -x+4-\frac{3}{x} = \infty$.

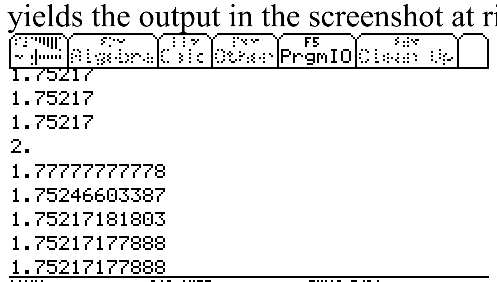
14. Show that the y-coordinate of the point (x,y) on the curve described by $\frac{x^2+1}{\ln\left(y^2+4\right)} = 1$ that is closest to the point (0,2) can be found by solving $2y^3-4y^2+10y-16=0$. Use Newton's method to solve the equation. *Hint*: Minimizing the distance D is equivalent to minimizing the square of the distance, D^2. Also, the curve has origin symmetry, so you can just work with the distance in the first quadrant.

SOLN: $\frac{x^2+1}{\ln\left(y^2+4\right)} = 1 \Leftrightarrow x^2 = -1+\ln\left(y^2+4\right)$. If (x,y) is a point on this curve then the square of the distance from (x,y) to (0,2) is $D^2 = (x-0)^2+(y-2)^2 = -1+\ln\left(y^2+4\right)+(y-2)^2$.

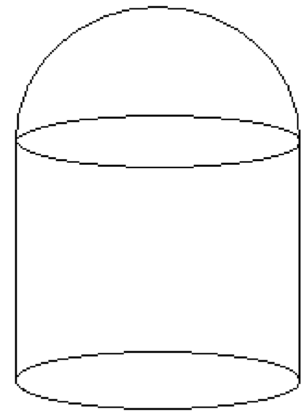
Differentiating, $2D\frac{dD}{dy} = \frac{2y}{y^2+4}+2(y-2)$, and setting the derivative to zero,

$\frac{2y+2(y-2)\left(y^2+4\right)}{y^2+4} = \frac{2y^3-4y^2+10y-16}{y^2+4} = 0$ and setting the numerator to zero yields the desired equation. On the TI Voyage 200, you can write a program for Newton's method (shown at left below). Calling "Newton(1)" yields the output in the screenshot at right.





15. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder with a hemispherical top. What radius and height will require the least amount of metal?



SOLN: The volume is fixed at $V = \pi r^2 h + \frac{2}{3}\pi r^3$. Solving for h:

$h = \frac{3V - 2\pi r^3}{3\pi r^2} = \frac{V}{\pi r^2} - \frac{2r}{3}$. Now the surface area is

$S = 2\pi rh + 2\pi r^2 = 2\pi r(h+r)$. Substituting for h in terms of r:

$$S = 2\pi rh + 2\pi r^2 = 2\pi r\left(\frac{V}{\pi r^2} + \frac{r}{3}\right) = \frac{2V}{r} + \frac{2\pi r^2}{3}$$

Setting the derivative to zero: $S'(r) = \frac{4\pi r}{3} - \frac{2V}{r^2} = 0 \Leftrightarrow 2\pi r^3 = 3V \Leftrightarrow \boxed{r = \sqrt[3]{\frac{3V}{2\pi}}}$