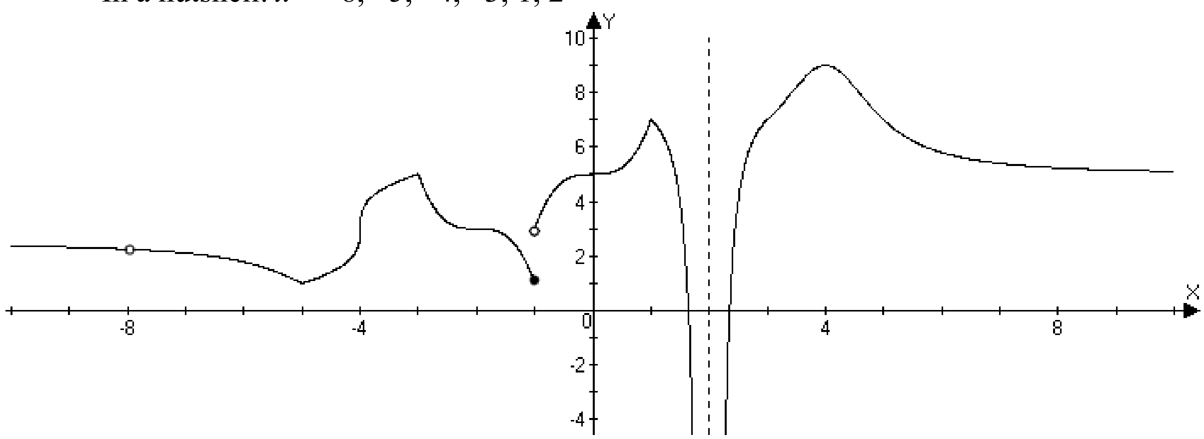
Math 1A – Final Exam Solutions – spring '07

1. The graph below shows $y = f(x)$.
   a. State, with reasons, the numbers a at which $\lim_{x\to a} f(x)$ does not exist.
   SOLN: $\lim_{x\to -1} f(x)$ does not exist since there's a jump discontinuity from $y = 1$ to $y = 3$ there. Also, $\lim_{x\to 2} f(x)$ does not exist since there's a vertical asymptote there.
   In a nutshell: $x = -1$ and 2
   b. State where the function is discontinuous and classify the discontinuity as removable, jump or asymptote.
   SOLN: In addition to the two discontinuities from (a) above, there's a removable discontinuity at $x = -8$.
   In a nutshell: $x = -8, -1$ and 2
   c. State, with reasons, the numbers at which the function is not differentiable.
   SOLN: The function is not differentiable where it's not continuous: $x = -8, -1$ and 2. It also has corners (not smooth places) at $x = -5, -3$ and 1.
   Finally it's not differentiable at $x = -4$ since the tangent line is vertical there.
   In a nutshell: $x = -8, -5, -4, -3, 1, 2$



2. Compute the limit, if it exists.
   a. $\lim_{x\to 2}\frac{x-2}{x^2-3x+2} = \lim_{x\to 2}\frac{x-2}{(x-2)(x-1)} = \frac{1}{2-1} = 1$
   b. $\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\left(\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}\right) = \lim_{h\to 0}\frac{h}{h\left(\sqrt{h+1}+1\right)} = \frac{1}{2}$
   c. $\lim_{x\to\infty}\frac{2x^4-x^2+4x}{16-x^4} = \lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{4}{x^3}}{\frac{16}{x^4}-1} = \frac{2-0+0}{0-1} = -2$
   d. $\lim_{\theta\to 0}\frac{\sin 5\theta}{2\theta} = \frac{5}{2}\lim_{5\theta\to 0}\frac{\sin 5\theta}{5\theta} = \frac{5}{2}$

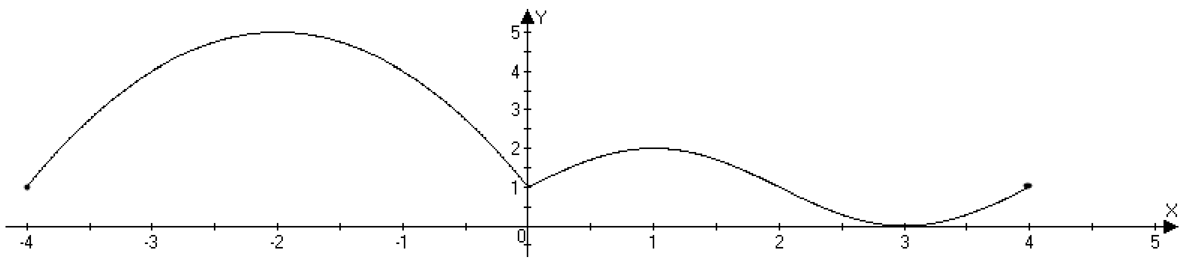
3. Use the definition of the derivative to derive one of the formula below (your choice)

SOLN: $\frac{d}{dx}\left(\sqrt{x}\right) \equiv \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$

$$\frac{d}{dx}(\sin x) \equiv \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} = 0+1*\cos x$$

4. Use the various shortcut rules (product rule, chain rule, etc.) to simplify the derivative:
   a. $f(t)=\frac{3t-5}{t^2+7t-11} \Rightarrow f'(t)=\frac{\left(t^2+7t-11\right)3-(2t+7)(3t-5)}{\left(t^2+7t-11\right)^2}=\frac{-3t^2+10t+2}{\left(t^2+7t-11\right)^2}$
   b. $g(x)=x^5\tan(2x) \Rightarrow g'(x)=5x^4\tan(2x)+2x^5\sec^2(2x)$
   c. $h(u)=\sin(\sin(\sin u)) \Rightarrow h(u)=\cos(\sin(\sin u))\cos(\sin u)\cos u$

5. Sketch a graph for a function defined on [-4,4] that meets the given conditions:
   $f(0)=1,\quad f'(-2)=f'(1)=f'(3)=0,\ f'(0)$ is undefined
   $f'(x)<0$ only on $(-2,0)\cup(1,3)$
   $f''(x)<0$ only on $(-4,0)\cup(0,2)$



6. Find a linear approximation for the function $R=\frac{100r}{100+r}$ near $a = 50$
   and use it to approximate R when $r = 53$.
   SOLN: $R'=\frac{(100+r)100-100r}{(100+r)^2}=\frac{10000}{(100+r)^2}$, So at $a = 50$, $R'=\frac{10000}{(150)^2}=\frac{4}{9}$. Thus the approximating line is $R=\frac{100}{3}+\frac{4}{9}(r-50)$. At $r = 53$, $R\approx\frac{100}{3}+\frac{4}{9}(53-50)=\frac{104}{3}$

7. Find an equation for the line tangent to $y=e^{\cos 2x}$ at $x=\frac{\pi}{4}$.
   SOLN: $y'=-2\sin 2xe^{\cos 2x}$ so at $x=\frac{\pi}{4}$, the slope of the tangent line is $m=-2$ and the equation for the tangent line is $y=1-2\left(x-\frac{\pi}{4}\right)$.

8. Under what condition(s) will the points on a tangent line lie beneath the curve?
SOLN: If the curve is concave up, at least in the immediate neighborhood of the point.

9. Find the local and global extreme values of the function $y = x^2 \ln x^2$ on the interval [-2,2].
SOLN: It may be a bit simpler to work with the equivalent formula, $y = 2x^2 \ln|x|$.
$y' = 4x\ln|x| + 2x = 0$ If x is not 0 then $\ln|x| = -\frac{1}{2} \Leftrightarrow x = \pm e^{-1/2} \approx \pm 0.6065$

10. Find the point on the parabola $x = 16 - y^2$ that is closest to the point (16, 3).
SOLN: The square of the distance from (16, 3) to the parabola is
$D^2 = (16-x)^2 + (3-y)^2 = (y^2)^2 + (3-y)^2$. Differentiating with respect to y we have
$2D\frac{dD}{dy} = 4y^3 - 2(3-y) = 4y^3 + 2y - 6 = 2(y-1)(2y^2+2y+3)$ so the only real value of y that assures $\frac{dD}{dy} = 0$ is $y = 1$, which means $x = 15$.

11. If $y = e^{xy}$, find an expression for $\frac{dy}{dx}$ and write a linearization for y near (0,1).
SOLN: First write the equivalent logarithmic equation: $\ln y = xy$, the differentiate with respect to x implicitly: $\frac{1}{y}\frac{dy}{dx} = y + x\frac{dy}{dx}$. Plugging in $(x, y) = (0, 1)$ we get $\frac{dy}{dx} = 1$. Thus the equation for the tangent line is $y = x + 1$.

12. Find an equation for the line tangent to $x^3 + x^2y + 2y^3 = 2$ at $(2,-1)$.
SOLN: Differentiating with respect to x implicitly: $3x^2 + 2xy + x^2\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 0$.
Plugging in the point coordinates, $12 - 4 + 4\frac{dy}{dx} + 6\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{4}{5}$, so the tangent line is the solution set to $y = -1 - \frac{4}{5}(x-2) = -\frac{4}{5}x + \frac{3}{5}$

13. Show that the y-coordinate of the point (x,y) on the curve described by $\frac{x^2+1}{\ln(y^2+4)} = 1$ that is closest to the point (0,2) can be found by solving $y^3 - 2y^2 + 5y - 8 = 0$. Use Newton's method to solve the equation. *Hint*: Minimizing the distance D is equivalent to minimizing the square of the distance, D^2. Also, the curve has origin symmetry, so you can just work with the distance in the first quadrant.
SOLN: Here the square of the distance from (0,2) to the curve is
$D^2 = (x-0)^2 + (y-2)^2 = x^2 + (y-2)^2$. On the curve, $x^2 = -1 + \ln(y^2+4)$. Substituting, for x^2, we get $D^2 = -1 + \ln(y^2+4) + (y-2)^2$. Differentiating with respect to y;
$2D\frac{dD}{dy} = \frac{2y}{y^2+4} + 2(y-2)$, and setting this to zero we have the equivalent equation

$y^3 - 2y^2 + 5y - 8 = 0$. To find a zero of this cubic, iterate

$$y_{n+1} = y_n - \frac{y_n^3 - 2y_n^2 + 5y_n - 8}{3y_n^2 - 4y_n + 5} = \frac{2y_n^3 - 2y_n^2 + 8}{3y_n^2 - 4y_n + 5}$$

The TI85 screen captures showing how this can be implemented are below (assuming this formula is in *y*4 on the graph page:

```
1→x
                       1
y4→x
                       2
           1.7777777778
           1.75246603387
           1.75217181803
```

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1.7777777778
1.75246603387
1.75217181803
1.75217177888
1.75217177888
1.75217177888
1.75217177888
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14. A box with a square base with no top has a surface area of 108 square feet. Find the dimensions that will maximize the volume.
SOLN: Let x = the side of the square base and let y = the height of the box. Then the surface area is $x^2 + 4xy = 108$ which means that the height of the box can be represented as $y = \frac{108 - x^2}{4x}$. The volume of the box is $V = x^2 y$. Substituting for y we have $V = x^2 y = \frac{108x - x^3}{4} = 27x - \frac{x^3}{4}$. Thus the instantaneous rate of change in the volume per change in x is $\frac{dV}{dx} = 27 - \frac{3x^2}{4} = 0 \Rightarrow x = 6 \Rightarrow y = 3$

15. Use Newton's method to approximate a solution to $x^2 = \cos 2x$ to 8 digits. Simplify the iteration formula for this case, use an initial value of $x = 1$ and show what the iterates are.
SOLN: $x^2 = \cos 2x \Leftrightarrow f(x) = x^2 - \cos 2x = 0$. Applying Newton's iteration formula we have $x_{n+1} = x_n - \frac{x_n^2 - \cos 2x_n}{2x_n + 2\sin 2x_n}$. The iterates starting at $x = 1$ are then

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                       1
y5→x
           .629144517598
           .601189561129
           .600769248799
            .60076914967
            .60076914967
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