Math 1A – Calculus – Chapter 4 Test Solutions – Spring 07

1. A bug is crawling along the curve $y = \sqrt[3]{x}$ so that as the bug passes through the point (27, 3) its x-coordinate increases at a rate of 0.2 cm/s.
   a. How fast is the y-coordinate increasing?

   SOLN: $\left.\frac{dy}{dt}\right|_{x=27} = \left.\frac{1}{3\sqrt[3]{x^2}}\frac{dx}{dt}\right|_{x=27} = \frac{1}{3\sqrt[3]{27^2}}(0.2) = \frac{1}{135} = 0.00\overline{740}$ cm/sec

   b. How distance from the bug to the origin changing at this instant?

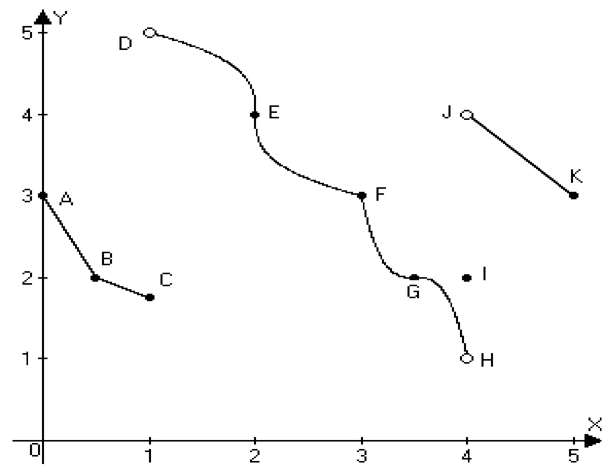
   SOLN: $D^2 = x^2 + y^2 \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dD}{dt} = \frac{27(0.2)+3\left(0.00\overline{740}\right)}{\sqrt{3^2+27^2}} \approx 0.19959$

2. Two sides of a triangle have lengths $b = 3$ and $c = 4$. What is the rate of change in the third side a per change in the angle between b and c when $a = 5$? Recall that the law of cosines involves the formula $a^2 = b^2 + c^2 - 2bc\cos\theta$.

   SOLN: Differentiating, we have $2a\frac{da}{d\theta} = 2bc\sin\theta \Leftrightarrow \frac{da}{d\theta} = \frac{bc\sin\theta}{a}$. Plugging in $b = 3$, $c = 4$ we see we have $\theta = \frac{\pi}{2}$ and $a = 5$ so that $\frac{da}{d\theta} = \frac{12}{5}$

3. Sketch the graph of a function on [0, 5] that has no global maximum nor minimum, two local minima, one local maximum and four critical numbers.

SOLN: Since the endpoints are going to be either local max or min we have to work in 4 critical numbers on the interior, while creating only one additional local extremum. For the function graphed at right, there is a local max at A, local minima at C and K and critical points at B, E, F and G (the function is continuous at B, E and F, but the derivative does not exist; at G there is a stationary point, but no max nor min.) There is no global extreme since the jump discontinuities from D to C and from H to I to J involve an unrealized max at D and an unrealized min at H.



4. Find the inflection points of $y = x^2 e^{3x}$.

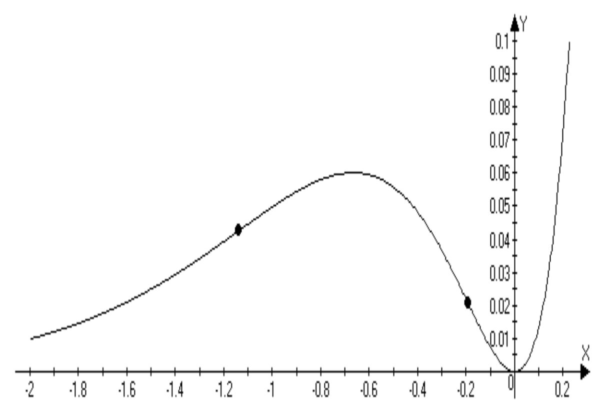
   SOLN: $y' = \left(2x+3x^2\right)e^{3x} \Rightarrow y'' = \left(2+12x+9x^2\right)e^{3x} = 0 \Leftrightarrow x = \frac{-12\pm 6\sqrt{2}}{18} = -\frac{2}{3}\pm\frac{\sqrt{2}}{3}$. That

means that $y = \left(\frac{2}{3}\mp\frac{4\sqrt{2}}{9}\right)\exp\left(-2\pm\sqrt{2}\right)$. Thus the inflection points are

$\left(-\frac{2}{3}-\frac{\sqrt{2}}{3}, \left(\frac{2}{3}+\frac{4\sqrt{2}}{9}\right)\exp\left(-2-\sqrt{2}\right)\right) \approx (-1.14, 0.043)$

and

$\left(-\frac{2}{3}+\frac{\sqrt{2}}{3}, \left(\frac{2}{3}-\frac{4\sqrt{2}}{9}\right)\exp\left(-2+\sqrt{2}\right)\right) \approx (-0.20, 0.021)$

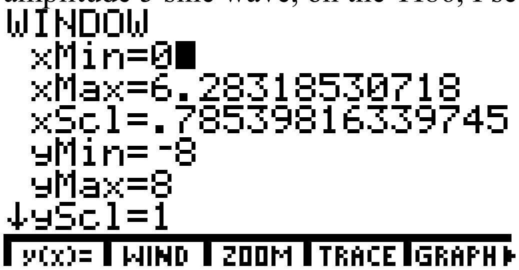


5. Suppose $f'(x)=(x+2)^2(x-7)^3(x-11)^4$. On what intervals is *f* increasing?

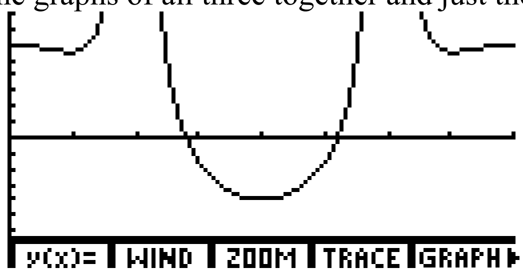
SOLN: The function *f* is increasing on the interval $(7,\infty)$.

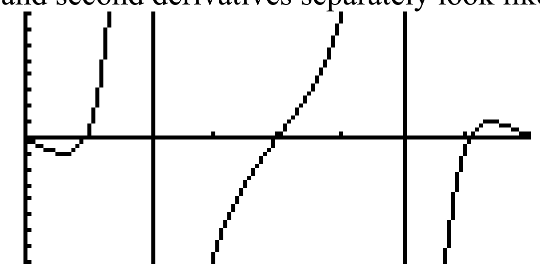
6. Use a calculator to examine the graphs of $f(x)=\tan x+5\sin x$ and $f'(x)$ and $f''(x)$.

SOLN: To get started, it's a good idea to find formulas for $f'(x)=\sec^2 x+5\cos x$ and $f''(x)=\dfrac{2\sin x}{\cos^3 x}-5\sin x$ and observe that each has a period of 2π. Given that we're adding to an amplitude 5 sine wave, on the TI86, I set up my Window page and defined the functions this way:

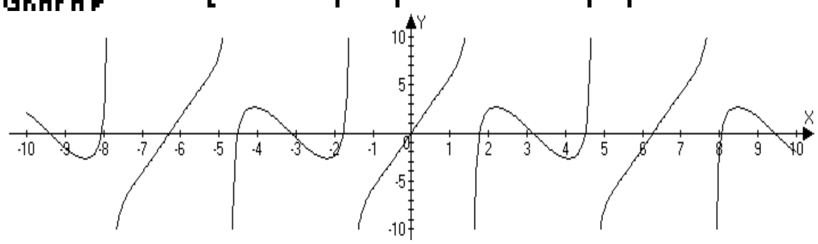


The graphs of all three together and just the first and second derivatives separately look like this:





The function itself looks like this:



a. How many local extrema are in each period? SOLN: Two, a max and a min.
b. How many intervals of increase are in each period?
   SOLN: 3 or 4 intervals of increase and 1 or 2 intervals of decrease (depending on how you set the period interval)
c. How many inflection points are in each period?
   SOLN: There are 3 inflection points in each period…as I've shown them, but there are would-be inflection points at the endpoints, so by shifting the window slightly you pick up a fourth.

7. Find $\lim\limits_{x\to 0}\dfrac{e^{5x}-1-\sin x}{x^2+x}$ using L'Hospital's rule, if appropriate.

SOLN: $\lim\limits_{x\to 0}\dfrac{e^{5x}-1-\sin x}{x^2+x}=\lim\limits_{x\to 0}\dfrac{5e^{5x}-\cos x}{2x+1}=4$

8. For what values of *a* and *b* is the following true: $\lim\limits_{x\to 0}\dfrac{\sin 6x}{x^3}+a+\dfrac{b}{x^2}=0$?

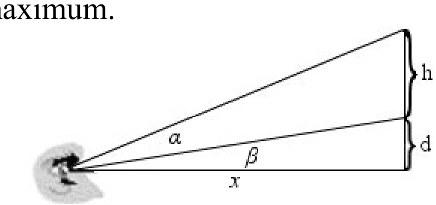
SOLN: $\lim\limits_{x\to 0}\dfrac{\sin 6x}{x^3}+a+\dfrac{b}{x^2}=a+\lim\limits_{x\to 0}\dfrac{bx+\sin 6x}{x^3}=a+\lim\limits_{x\to 0}\dfrac{b+6\cos 6x}{3x^2}=a+\lim\limits_{x\to 0}\dfrac{b+6\cos 6x}{3x^2}$

Now for this limit to exist, we need the numerator to approach zero together with the denominator. Thus *b* = –6 and we have $a+\lim\limits_{x\to 0}\dfrac{-6+6\cos 6x}{3x^2}=a+\lim\limits_{x\to 0}\dfrac{-36\sin 6x}{6x}=a-36=0$ so *a* = 36.

9. If $y = x^2 - x + 1$, what value will minimize the product xy on the interval [0, 2] ?
SOLN: We seek to minimize $x^3 - x^2 + x$ whose rate of change per change in x is $3x^2 - 2x + 1$. This quadratic has a negative discriminant, so it has no real zeros and is, in fact, always positive. Thus the product is always increasing and its minimum is at the left endpoint where $xy = 0$.

10. Find the points on the ellipse $9x^2 + y^2 = 9$ which are farthest from the point (0, 3).
SOLN: We seek to maximize the distance from (0,3) to (cos□, 3sin□). The square of this distance is $D^2 = (0-\cos\theta)^2 + (3-3\sin\theta)^2 = \cos^2\theta + 9 - 18\sin\theta + 9\sin^2\theta = 10 - 18\sin\theta + 8\sin^2\theta$.
Thus $2D\frac{dD}{d\theta} = -18\cos\theta + 16\sin\theta\cos\theta = 0 \Rightarrow \cos\theta = 0$ or $\sin\theta = \frac{9}{8}$. The sin function can't be greater than 1, so it must be that $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Now $\theta = \frac{\pi}{2}$ corresponds to the point (0, 3), which is zero distance (a minimum) and $\theta = \frac{3\pi}{2}$ corresponds to (0, –3), at the extreme opposite end of the ellipse (a distance 6 away) which must be a maximum.

11. A painting in an art gallery has height h and is hung so that the lower edge is a distance d above the eye of the observer. How far from the wall should the viewer stand so as to maximize the angle α the painting subtends at the viewer's eye?



SOLN: Let x = the distance to the wall and let β be the angle the line from the eye to the lower edge of the picture forms with the horizontal. Then $\tan\beta = \frac{d}{x}$ and

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\tan\alpha + \frac{d}{x}}{1 - \frac{d}{x}\tan\alpha} = \frac{x\tan\alpha + d}{x - d\tan\alpha} = \frac{h+d}{x}.$$

The last equation here relates variables x and α. Clearing out the fractions we get the equivalent equation,
$x^2\tan\alpha + dx = (h+d)(x - d\tan\alpha)$. Assuming that x is a function of α we differentiate with respect to α: $2x\frac{dx}{d\alpha}\tan\alpha + x^2\sec^2\alpha + d\frac{dx}{d\alpha} = (h+d)\left(\frac{dx}{d\alpha} - d\sec^2\alpha\right)$. Grouping terms we have:
$(h - 2x\tan\alpha)\frac{dx}{d\alpha} = \left(d(h+d) + x^2\right)\sec^2\alpha$. Thus if $\frac{dx}{d\alpha} = 0$, then
$d(h+d) + x^2 = 0 \Rightarrow \boxed{x = \sqrt{d(h+d)}}$, the geometric mean of the lift with total height, of course!

12. Use Newton's method to approximate $\sqrt[3]{1729}$ accurate to 8 decimal places. Start with $x_1 = 8$ and show the iteration formula and the iterates up to convergence.
SOLN: Let $f(x) = x^3 - 1729$. Then f has a zero at $\sqrt[3]{1729}$ and we iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 1729}{3x_n^2} = \frac{2x_n^3 + 1729}{3x_n^2}$$

```
8→x
                  8
(2 x^3+1729)/(3*x^2)→
x
          14.3385416667
          12.3622903151
          12.0126947196
█
```

```
x
          14.3385416667
          12.3622903151
          12.0126947196
          12.0023233357
          12.0023143684
          12.0023143684
█
```